PAPER I

ALGEBRA SUMMARY

① FACTORS

DOTS : $x^2 - y^2 = (x-y)(x+y)$

SOTC : $x^3 + y^3 = (x+y)(x^2 - xy + y^2)$

DOTC : $x^3 - y^3 = (x-y)(x^2 + xy + y^2)$

PS : $x^2 \pm 2xy + y^2 = (x \pm y)^2$

(Always take out the HCF first.)

② POWERS / ROOTS

Know the rules

eg $a^m . a^n = a^{m+n}$ etc

$\sqrt{ab} = \sqrt{a}\sqrt{b}$ and $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$ (see p 11 Bk 1)

③ LOGS

$$\boxed{\log_a x = y \iff x = a^y}$$

Know the rules

eg $\log_a m + \log_a n = \log_a (mn)$

$\log_a m = \log_b m \times \log_a b$ etc. (see p18 Bk 1)

④ IDENTITIES

These are true for all values of the variables.

2 methods $\longrightarrow$ (i) Sub in "nice" values

eg $x - 1 = p(x+2) + q(x-3)$

let $x = -2 \Rightarrow -3 = -5q \Rightarrow q = \frac{3}{5}$

let $x = 3 \Rightarrow 2 = 5p \Rightarrow p = \frac{2}{5}$

→ (ii) Compare coefficients

eg $2x^2 + 8x + 5 = 2(x+A)^2 + B$

$\Rightarrow 2x^2 + 8x + 5 = 2x^2 + 4Ax + 2A^2 + B$

x-terms $\Rightarrow 4A = 8 \Rightarrow A = 2$

constant terms $\Rightarrow 2A^2 + B = 5 \Rightarrow B = -3$

5 FACTORIALS

(i) $n! = n(n-1)(n-2)\ldots 2.1$

(ii) $0! = 1$

(iii) Be able to $\div, \times, +, -$ factorials

eg $10! + 11! = 10!(1+11) = 12.10!$

6 BINOMIAL COEFFS

(i) $\binom{n}{r} = {}^nC_r = \dfrac{n!}{r!(n-r)!}$ need to know this formula for some Q's.

(ii) Quick way: $\binom{n}{r} = \dfrac{r \text{ numbers starting from } n}{r!}$

eg $\binom{5}{3} = \dfrac{5.4.3}{3.2.1}$

(iii) $\binom{n}{r} = \binom{n}{n-r}$ eg $\binom{10}{3} = \binom{10}{7}$

(iv) $\binom{n}{0} = 1$ and $\binom{n}{1} = n$

7 EQUATIONS

(a) FACTOR THEOREM

If $(x-k)$ is a factor of $f(x)$, then k is a root of $f(x) = 0$, and vice versa.

(b) Roots = places curve crosses X-axis
(Remember they "work" in the eqn)

(c) LINEAR EQNS → easy

(d) QUADRATIC EQNS

(i) $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

(ii) Properties of Roots : If $ax^2 + bx + c = 0$

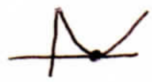
$\text{Sum} = \alpha + \beta$

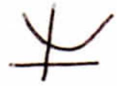
$\text{Product} = \alpha\beta$

(ii) Form quadratic given roots :

$x^2 - Sx + P = 0$

(iii) Real roots $\Rightarrow$ $b^2 \geq 4ac$ 

Equal roots $\Rightarrow$ $b^2 = 4ac$ 

Complex roots $\Rightarrow$ $b^2 < 4ac$ 

(e) CUBIC EQNS

(i) To solve : guess one root, then long ÷ to find other 2 roots.

(ii) Properties of Roots : If $ax^3 + bx^2 + cx + d = 0$

$\text{Sum} = \alpha + \beta + \gamma$

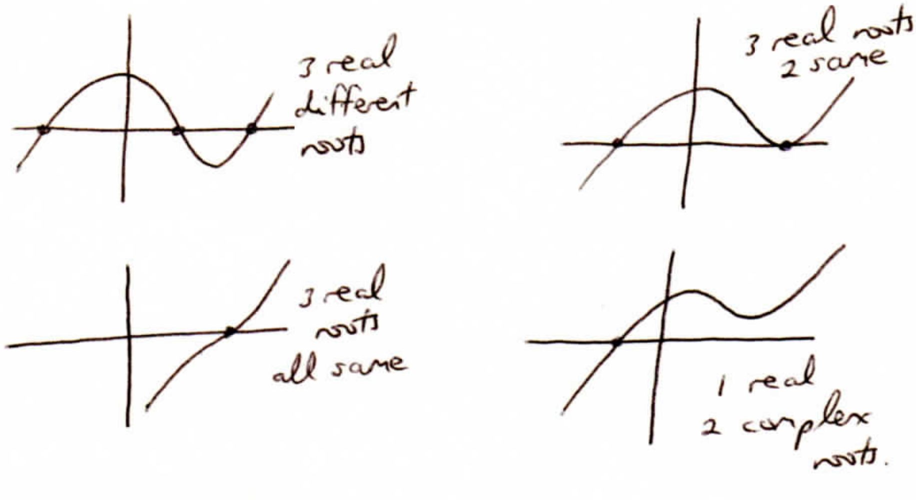
$\text{Sum of Product in Pairs} = \alpha\beta + \beta\gamma + \alpha\gamma$

$\text{Product} = \alpha\beta\gamma$

(iii) Form cubic given roots:

$$x^3 - Sx^2 + SPPx - P = 0$$

(iv)



(f) LITERAL EQNS (letters)

(i) get rid of fractions first (common denom.)

(ii) To get rid of $\sqrt{\ }$, isolate it & square both sides

(g) MODULUS EQNS

(i) $|x| = 10 \Rightarrow x = \pm 10$

(ii) $|2x+1| = |3x+2| \Rightarrow 2x+1 = \pm(3x+2)$ etc.

(iii) $\left|\sqrt{2x+3}\right| = 5 \Rightarrow 2x+3 = 25$ etc (square both sides)

(h) EXPONENT EQNS $(a^{f(x)})$

(i) Get everything with same base number

eg $2^6 = 4^x \Rightarrow 2^6 = 2^{2x} \Rightarrow x = 3$

(ii) Hard ones:

(a) If no common base number, take $\log_{10}$ of both sides

eg $\dfrac{7^x}{2^{2x}} = 11 \Rightarrow \dfrac{7^x}{4^x} = 11$

$\Rightarrow \left(\dfrac{7}{4}\right)^x = 11 \Rightarrow x \log_{10}\left(\dfrac{7}{4}\right) = \log_{10} 11$

etc.

(iii) 3 linear, 3 unknowns
→ eliminate 1 letter using 2 eqns
→ " same letter using 2 others
→ solve 2 new eqns as normal
→ sub back in etc.

(iv) 2 log eqns, 2 unknowns
→ eliminate logs with same base
→ get out of logs to find x
→ Sub back in to find y.

⑧ INEQUALITIES

(a) Same as equations except
NEVER MULTIPLY OR DIVIDE BY NEGATIVE NO.

(b) LINEAR → easy

(c) QUADRATICS : $ax^2 + bx + c \leq 0$
→ get all terms on one side, zero on other
→ solve corresponding eqn. to get roots α, β
→ put α & β in ascending order
→ test regions with nice numbers
$\longleftarrow \alpha \longleftrightarrow \beta \longrightarrow$

(d) RATIONALS : $\frac{p(x)}{q(x)} < 0$

→ Simplify if possible
→ Multiply across by (common denominator)^2
unless you're sure it's positive.
→ Solve as before.

(e) MODULUS : $|ax + b| \geq c$
→ Solve corresponding equation
→ Do region test on roots $\longleftarrow \alpha \longleftrightarrow \beta \longrightarrow$

COMPLEX NOS. SUMMARY

① REMEMBER $i = \sqrt{-1}$

EVERY complex no. is of the form $Re + i.Im$.

② To work out a power of i, divide the power by 4 and just take the remainder

eg $i^{37} = i^1 = i$ (since $i^4 = 1$)

③ To $+, -, \times$ complex nos, just do like normal algebra, but remember $\boxed{i^2 = -1}$

④ CONJUGATE : $\overline{a+bi} = a - bi$

eg $\overline{4+3i} = 4-3i$, $\overline{-2-7i} = -2+7i$ etc.

Remember : $(a+ib)(a-ib) = a^2 + b^2$

⑤ DIVISION : Multiply top & bottom by conjugate of bottom

eg $\dfrac{3-4i}{2+3i} = \dfrac{3-4i}{2+3i} \times \dfrac{2-3i}{2-3i} = \dfrac{6-8i-9i-12}{4+9}$

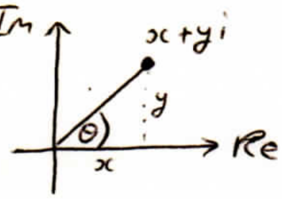
$= \dfrac{-6}{13} - \dfrac{17}{13}i$

⑥ SQUARE ROOTS :

eg find $\sqrt{z}$: where $z = a+bi$

Let answer $= c+di$, square both sides & solve the equation.

(7) ARGAND DIAGRAM :



(8) MODULUS (Distance from origin)

$$|x + yi| = \sqrt{x^2 + y^2}$$

(9) ARGUMENT ($\arg z$)

$$\theta = \arg z = \tan^{-1}\left(\frac{Im}{Re}\right) = \tan^{-1}\left(\frac{y}{x}\right)$$

ie $\tan\theta = \frac{Im}{Re}$

HINT : Draw a sketch to see where θ is (ie which quadrant). Then use 

etc.

(10) EQUATIONS :

- Tidy up both sides as much as possible.
- Let $Re = Re$ and $Im = Im$ on the 2 sides.
- Solve the resulting simultaneous eqns.

NB Read carefully to see if there is any $\bar{z}$ in the question or if it's all just z.

(11) PROVING PROPERTIES : eg Show $|zw| = |z||w|$

Treat like trig identities

ie don't cross over from 1 side to other

Just show $LHS = RHS$ separately.

(12) QUADRATIC & CUBIC EQNS :

(a) QUADRATIC :

$$z = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \alpha, \beta$$

Sum $= \alpha + \beta$. Prod $= \alpha\beta$

Form quadratic $\Rightarrow$ $z^2 - Sz + P = 0$

- Just like algebra except b^2-4ac will be < 0.
- If a complex no. is a root of an equation with all <u>real</u> <u>coefficients</u>, then so is its complex conjugate. (applies for cubics also)

(b) CUBICS :

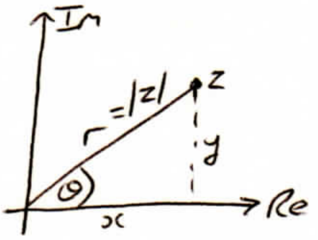
$S = \alpha + \beta + \gamma$

$SPP = \alpha\beta + \beta\gamma + \alpha\gamma$

$P = \alpha\beta\gamma$

Form cubic $\Rightarrow$ $z^3 - Sz^2 + SPPz - P = 0$

(13) POLAR FORM



$z = x + iy$ ← Cartesian

$\Rightarrow z = r(\cos\theta + i\sin\theta)$ ← Polar

r: $|z|$, θ: $\arg z$

To change to Polar :

(i) Find $r = |z| = \sqrt{Re^2 + Im^2}$

(ii) Draw a picture to see what quadrant.

(iii) Find θ using a bit of trig & picture

$$\left(\tan\theta = \frac{Im}{Re}\right)$$

(iv) Write answer $z = r(\cos\theta + i\sin\theta)$

(v) For general polar form write

$$z = r\left(\cos(\theta + 2n\pi) + i\sin(\theta + 2n\pi)\right)$$

To change from Polar to Cartesian

Just work out $\cos\theta$ and $\sin\theta$ and multiply out the bracket.

(14) DE MOIVRE

$$(\cos\theta \pm i\sin\theta)^p = \cos p\theta \pm i\sin p\theta \quad \text{for all } p \in \mathbb{R}.$$

• Proof is needed (for $p \in \mathbb{N}_0$). → see INDUCTION.

(a) STEPS FOR WHOLE NUMBER POWERS :

(i) Write z in polar form

(ii) Use de Moivre (ie multiply angle by power)

(iii) Change back to Cartesian form.

(b) STEPS FOR FRACTION POWERS (ROOTS) :

(i) Write z in GENERAL POLAR FORM (ie with $2n\pi$)

(ii) Use de Moivre

(iii) Sub in values for n (ie $n=0, n=1, n=2$ etc)

You need as many answers as the number on the bottom of the power eg $z^{\frac{1}{4}} \Rightarrow 4$ answers/roots.

(iv) Change each answer back to Cartesian form for nice angles.

(c) PROOFS OF TRIG IDENTITIES :

(i) Write down de Moivre for the number in the multiple angle

eg Prove $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

Soln $(\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta$

etc.

(ii) Expand out the bracket

(iii) Put Re = Re and Im = Im

(iv) Look for answers needed.

(15) ROTATIONS & DILATIONS

(a) When you multiply a complex no. by i it rotates 90° anticlockwise. → rotation

(b) When you multiply a complex no. by a real number, k, its modulus gets multiplied by k (ie further from origin) → dilation

(c) When you multiply a complex no. by another complex no. it causes rotation and dilation (ie spiral outwards or spiral inwards)

SEQUENCES & SERIES.

① GENERAL TERM : T_n / u_n

Just sub in whole numbers for n to get each term.

② SUM OF TERMS : S_n

- If they give you the formula just sub in for n.
- NB $S_2 - S_1 = T_2$, $\boxed{S_n - S_{n-1} = T_n}$

③ CONVERGENCE / DIVERGENCE

If $\lim_{n \to \infty} T_n =$ finite number $\Rightarrow$ CONVERGENT

If $\lim_{n \to \infty} T_n = \infty$ $\Rightarrow$ DIVERGENT

④ ARITHMETIC SEQUENCES (AP)

(i) Adding on same number each time

(ii) $a, a+d, a+2d, \ldots$

$a = 1^{st}$ term

$d =$ common difference

(iii) $\boxed{T_n = a + (n-1)d}$ = n^{th} term

(iv) If $T_{n+1} - T_n =$ constant for all n

then sequence is an AP.

(v) $\boxed{S_n = \frac{n}{2}\{2a + (n-1)d\}}$ = sum of 1^{st} n terms.

(vi) If question talks about 3 consecutive terms in an AP, call them

$a-d, a, a+d$

(vii) If you're told a, b, c are in AP then

$$b - a = c - b$$

(viii) The <u>arithmetic mean</u> of 2 numbers, a and b, is just $\frac{a+b}{2} = m$. Then a, m, b are in AP.

⑤ GEOMETRIC SEQUENCES (GP)

(i) Multiplying by same number each time.

(ii) $a, ar, ar^2, ar^3, \ldots$ $a = 1^{st}$ term, $r =$ common ratio

(iii) $\boxed{T_n = ar^{n-1}}$ = n^{th} term

(iv) If $\frac{T_{n+1}}{T_n}$ = constant for all n then sequence is a GP (ie $\frac{\text{ANY TERM}}{\text{PREVIOUS TERM}}$)

(v) $\boxed{S_n = \frac{a(1-r^n)}{1-r}}$ = sum of 1^{st} n terms

(vi) If question talks about 3 consecutive terms in a GP, call them

$$\frac{a}{r}, a, ar$$

(vii) If you're told a, b, c are in GP then

$$\frac{b}{a} = \frac{c}{b}$$

(viii) If $|r| < 1$ then $\boxed{S_\infty = \frac{a}{1-r}}$ (proof required)

(*) Ex Write $0.2\dot{5}$ as a rational (fraction)

Soln $0.255555\ldots = \frac{2}{10} + \frac{5}{100} + \frac{5}{1000} + \frac{5}{10000} + \ldots$

$$= \frac{2}{10} + \frac{5}{100}\left(1 + \frac{1}{10} + \frac{1}{100} + \ldots\right)$$

Bracket is a GP, $a = 1$, $r = \frac{1}{10}$

$$S_\infty = \frac{a}{1-r} = \frac{1}{1-\frac{1}{10}} = \frac{10}{9}$$

$$\therefore 0.2\dot{5} = \frac{2}{10} + \frac{5}{100}\left(\frac{10}{9}\right) = \frac{2}{10} + \frac{5}{90} = \boxed{\frac{23}{90}}$$

⑥ SIGMA NOTATION Σ

Means add up

eg $\sum_{r=1}^{4}(2r+1) = (2(1)+1) + (2(2)+1) + (2(3)+1) + (2(4)+1)$

⑦ BINOMIAL THEOREM

p20 tables

$$(x+y)^n = \binom{n}{0}x^n y^0 + \binom{n}{1}x^{n-1}y^1 + \ldots + \binom{n}{r}x^{n-r}y^r + \ldots + \binom{n}{n}x^0 y^n$$

Be able to use this to multiply out brackets quickly.

FINANCIAL MATHS

① Know how to do.....

(a) RATIO & PROPORTION

(b) PERCENTAGES

(c) FOREIGN EXCHANGE

(d) SCIENTIFIC NOTATION

(e) RELATIVE ERROR / % ERROR

(f) TOLERANCE & WASTAGE

② INCOME TAX

Gross ⇒ before tax taken off

Net ⇒ after tax etc taken off

Tax payable = gross tax − tax credit

③ COMPOUND INTEREST

$$F = P(1+i)^t$$

final amount

principal (start)

i = interest rate as a decimal

t = time

AER = EAR = CAR

= Annual equivalent rate

= % interest over the whole year

$F = Pe^{it}$ → formula for CONTINUOUS COMPOUNDING

④ DEPRECIATION

$$F = P(1-i)^t$$

$$F = Pe^{-it} \rightarrow \text{CONTINUOUSLY COMPOUNDED DEPRECIATION}$$

where F = final value = Net Book Value

⑤ PRESENT VALUE **

= amount you need to invest now to earn a certain amount at a certain time in the future

ie $$P = \frac{F}{(1+i)^t}$$

comes from compound interest formula backwards.

⑥ NET PRESENT VALUE

Used to see if something is a good investment or not

NPV = present value of all incomes − present value of all expenses.

NPV > 0 ⇒ good investment

⑦ ANNUITIES

= series of periodic equal payments for a specified term

eg pension, rent payments etc.

- If payments are not annual then either

  (i) Don't change AER, but use fractional amounts of time

  (ii) Adjust AER and use whole number amounts of time.

  eg 3.8% AER

  $\Rightarrow \quad 1.038 = 1\,(1+i)^{12}$

  $\therefore \quad \sqrt[12]{1.038} = 1+i$

  $\therefore \quad i = 0.00311$ per month

- Look out for Geometric Series when adding up future / present values of payments.

⑧ REGULAR SAVINGS

Same method as annuities.

⑨ BONDS

Fair market value for a bond

= present value of all future repayments to bond-holder.

(10) AMORTISED LOANS

Involves paying back a fixed amount at regular intervals over a fixed period of time (eg mortgage)

(p^{31} tables)

$$\boxed{A = \frac{Pi(1+i)^t}{(1+i)^t - 1}}$$ = annual repayment

where t = no. of years

i = annual rate as a decimal

(adjust t to months and i to monthly rate if repayments made monthly)

FUNCTIONS

① A function gives one unique output to each input.

② Be able to draw graphs of these functions:

(i) Linear $(y = ax + b)$

(ii) Quadratic $(y = ax^2 + bx + c)$

(iii) Cubic $(y = ax^3 + bx^2 + cx + d)$

(iv) Exponential $(y = ab^x)$ or $(y = e^{ax})$

(v) Trig $(y = a\sin bx$ etc$)$

(vi) Log $(y = \log_a x$ or $y = \ln x)$

③ Be able to use the graphs to answer questions

eg $f(x) = g(x)$ means point of intersection of the two graphs.

eg $f(x) < g(x)$ means graph of $f(x)$ is below graph of $g(x)$ etc.

④ Composite functions:

$$g \circ f(x) \qquad gf(x) \qquad g[f(x)]$$

These all mean "do function f first, then do g to your answer"

Ex $f(x) = 3x + 2$ $\qquad$ $g(x) = x^2$

$$\therefore \; g \circ f(x) = g(3x+2) = (3x+2)^2 = 9x^2 + 12x + 4$$

(7) SHIFTING & SCALING

Shifting : eg $f(x) = 3x^2$, $g(x) = 3x^2 + 6$ } graph "shifts" <u>up</u> by 6 units

or

$f(x) = x^2 + 3$, $g(x) = (x+2)^2 + 3$ } graph "shifts" <u>left</u> by 2 units

→ size & shape of graph don't change but position does.

Scaling : eg $f(x) = x^2$, $g(x) = 3x^2$ } shape of graph has changed (steeper)

(8) $y = x^2 + 6x + 8$ ⇒ y intercept is $(0, 8)$

$y = (x+4)(x+2)$ ⇒ roots $x = -4$ and $x = -2$

$y = (x+3)^2 - 1$ ⇒ local min. at $(-3, -1)$

LIMITS

① ALGEBRA

(i) Try filling in the number

(ii) If $\frac{0}{0}$ or $\frac{\infty}{\infty}$ then tidy up algebra

(eg factorise, cancel down)

(iii) Fill in number again

Ex
$$\lim_{x \to 5} \frac{x^2-25}{x-5} = \frac{5^2-25}{5-5} = \frac{0}{0} \times$$

$$\therefore \lim_{x \to 5} \frac{\cancel{(x-5)}(x+5)}{\cancel{x-5}} = \lim_{x \to 5} (x+5) = 5+5 = \boxed{10}$$

② INFINITY

(i) Fill in ∞

(ii) If answer is ∞, then there is NO LIMIT.

(iii) If answer is $\frac{\infty}{\infty}$ then

(a) Take out highest power of x top and bottom

(b) Cancel down

(c) Fill in ∞ again

Ex
$$\lim_{x \to \infty} \frac{2x^2+1}{3x^2+x} = \frac{\infty}{\infty} \times$$

$$\therefore \lim_{x \to \infty} \frac{\cancel{x^2}\left(2+\frac{1}{x^2}\right)}{\cancel{x^2}\left(3+\frac{1}{x}\right)} = \frac{2+\frac{1}{\infty}}{3+\frac{1}{\infty}} = \boxed{\frac{2}{3}}$$

③ EXPONENTIAL

Ex $\lim_{x \to \infty} 2\left(\frac{1}{3}\right)^x = 0$

Ex $\lim_{x \to \infty} 2(3)^x = \infty \quad \therefore$ NO LIMIT

Ex $\lim_{x \to \infty} 2\left(-\frac{1}{3}\right)^x = 0$

④ TRIG (Not officially on the course.... but good for pub quizzes!)

$$\boxed{\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1} \qquad \boxed{\lim_{\theta \to 0} \frac{\theta}{\sin\theta} = 1}$$

If θ in rads.

Ex
$$\lim_{\theta \to 0} \frac{\sin 4\theta}{7\theta}$$

$$= \lim_{\theta \to 0} \frac{\sin 4\theta}{4\theta} \cdot \frac{4}{7} = (1)\left(\frac{4}{7}\right) = \boxed{\frac{4}{7}}$$

$$\boxed{\lim_{\theta \to 0} \frac{\tan\theta}{\theta} = 1} \qquad \boxed{\lim_{\theta \to 0} \cos\theta = 1}$$

DIFFO

① WHAT IS IT ? → Finding the SLOPE of the TANGENT to a curve at a particular point.

$$\frac{dy}{dx} = f'(x) = m = \text{slope of tangent} = \text{derivative of } y$$

$= \text{rate of change of } y \text{ with respect to } x.$

② FIRST PRINCIPLES

Need to be able to do $y = ax + b$ and $y = ax^2 + bx + c$ type functions [see proofs sheets]

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

③ RULES

(a) $y = x^p \Rightarrow \frac{dy}{dx} = px^{p-1}$

PROD RULE (b) $y = (1^{st})(2^{nd}) \Rightarrow \frac{dy}{dx} = (1^{st})(\text{Diff } 2^{nd}) + (2^{nd})(\text{Diff } 1^{st})$

QUOTIENT RULE (c) $y = \frac{\text{TOP}}{\text{BOTTOM}} \Rightarrow \frac{dy}{dx} = \frac{(\text{Bottom})(\text{Diff Top}) - (\text{Top})(\text{Diff Bottom})}{(\text{Bottom})^2}$

CHAIN RULE (d) $y = f(u(x)) \Rightarrow \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

eg $y = (\text{STUFF})^p \Rightarrow \frac{dy}{dx} = p\,(\text{STUFF})^{p-1}(\text{Diff STUFF})$

④ ALGEBRA DIFFO

$$y = x^p \Rightarrow \frac{dy}{dx} = px^{p-1}$$

$$y = [f(x)]^p \Rightarrow \frac{dy}{dx} = p[f(x)]^{p-1} . f'(x)$$

⑤ TRIG DIFFO

(Use p25 tables)

Remember to diffo the angle as well!

eg $$y = \sin(x^2) \Rightarrow \frac{dy}{dx} = \cos(x^2) . (2x) = 2x\cos(x^2)$$

diffo sin — same angle — diffo angle

NB Angles are always in radians for trig diffo.

⑥ INVERSE TRIG DIFFO

(a) $$y = \sin^{-1}\frac{x}{a} \Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{a^2 - x^2}}$$ (p 25 tables)

$$y = \sin^{-1} f(x) \Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-(f(x))^2}} . f'(x)$$ (ie chain rule)

take a to be 1 — take x to be all of f(x) — diff "stuff inside"

(b) $$y = \tan^{-1}\frac{x}{a} \Rightarrow \frac{dy}{dx} = \frac{a}{a^2 + x^2}$$ (p25)

$$y = \tan^{-1} f(x) \Rightarrow \frac{dy}{dx} = \frac{1}{1+(f(x))^2} . f'(x)$$

take a to be 1 — diff stuff inside.

⑦ EXP DIFFO

(a) $y = e^x \Rightarrow \frac{dy}{dx} = e^x$ (p25)

$y = e^{f(x)} \Rightarrow \frac{dy}{dx} = e^{f(x)} . f'(x)$

same power   diffo the power

eg $y = e^{x^2} \Rightarrow \frac{dy}{dx} = e^{x^2} . 2x$

(b) $y = a^x \Rightarrow \frac{dy}{dx} = a^x \ln a$ (p25)

$y = a^{f(x)} \Rightarrow \frac{dy}{dx} = a^{f(x)} . \ln a . f'(x)$

same power   diffo the power

eg $y = 3^{6x^2} \Rightarrow \frac{dy}{dx} = 3^{6x^2}(\ln 3)(12x)$

⑧ ln DIFFO

$y = \ln x \Rightarrow \frac{dy}{dx} = \frac{1}{x}$ (p25)

$y = \ln f(x) \Rightarrow \frac{dy}{dx} = \frac{1}{f(x)} . f'(x)$

eg $y = \ln(3x^2+2) \Rightarrow \frac{dy}{dx} = \frac{1}{3x^2+2}(6x)$

"one over stuff"   "diff stuff"

NB $\boxed{\ln e^x = x \quad \text{and} \quad e^{\ln x} = x}$

9. HIGHER ORDER DIFFO & DIFFO EQNS

- $y = f(x)$, $\frac{dy}{dx} = f'(x)$ = diffo once

  $\frac{d^2y}{dx^2} = f''(x)$ = diffo twice

  etc

- Always tidy up answer before you diffo again.
- For diffo eqns, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ and sub into eqn to show it works

  eg If $y = xe^{-x}$, show that

  $$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0$$ (D.I.Y.!)

10. DIFFO APPLICATIONS

(a) Remember $\frac{dy}{dx}$ = slope of tangent

Tangents are straight lines so find slope & point.

Then $y - y_1 = m(x - x_1)$

(b) Curve increasing $\Rightarrow \frac{dy}{dx} > 0$

Curve decreasing $\Rightarrow \frac{dy}{dx} < 0$

Turning point $\Rightarrow \frac{dy}{dx} = 0$ (ie l.max or l.min)

Local max $\Rightarrow \dfrac{d^2y}{dx^2} < 0$

Local min $\Rightarrow \dfrac{d^2y}{dx^2} > 0$

Point of inflection $\Rightarrow \dfrac{d^2y}{dx^2} = 0$

Crosses X-axis $\Rightarrow$ $y = 0$

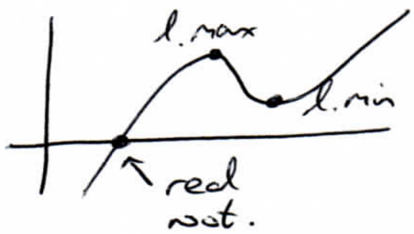
Crosses Y-axis $\Rightarrow$ $x = 0$

(c) Only fill in values for x and y after you've done the diffs.

(d) Be able to use diffs to sketch quadratic and cubic curves.
ie find l.max / l.min / PI and sketch

NB
For cubics, if l.max and l.min on same side of X-axis $\Rightarrow$ only ONE REAL ROOT

eg



(e) Be able to match a function's graph with the graph of the "slope function".
Look out for where slope = 0 ie l.max / l.min
and where slope is increasing / decreasing.

(f) FINDING MAX / MIN OF AREAS & VOLUMES

(i) Label everything in diagram

(ii) Write down formula for quantity to be max / min [Look up tables book]

(iii) Get formula in terms of only 1 letter.

(iv) Diff ... let answer = 0 ... find variable

(v) Don't forget to sub back in to find max or min area/volume if required.

(g) RATES OF CHANGE

(i) DISPLACEMENT (distance) / VELOCITY / ACCELERATION

- If x = distance / height / displacement

  then

  $$v = \frac{dx}{dt} = \text{velocity} = \text{rate of change of distance}$$

  $$a = \frac{d^2x}{dt^2} = \frac{dv}{dt} = \text{acceleration} = \text{rate of change of velocity}$$

- Careful with letters.... sometimes distance is called "s"

- Make sure you fill the numbers into the correct formula for the correct letter.
  .... read question carefully!

(ii) RELATED RATES OF CHANGE

- Remember chain rule

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

- Ex.

If radius of sphere is increasing at 3m/s, find rate of change of volume when radius is 12m.

$$\frac{dr}{dt} = 3 \qquad \frac{dV}{dt} = ? \qquad V = \frac{4}{3}\pi r^3$$

$$\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt}$$

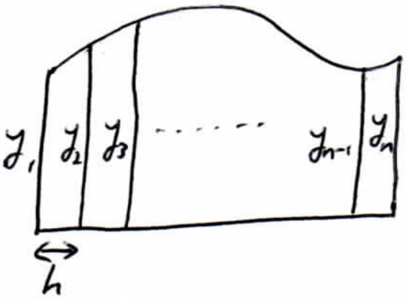
$$= (4\pi r^2)(3)$$

$$\left.\frac{dV}{dt}\right|_{r=12} = 4\pi (12)^2 (3) = \boxed{1728 \ m^3/s}$$

INTEGRATION

① TRAPEZOIDAL RULE

(p12 tables)



$$A = \frac{h}{2}\left[y_1 + y_n + 2\left(y_2 + y_3 + \dots + y_{n-1}\right)\right]$$

1st height (might be 0)   last height (might be 0)   all other heights

② ANTI-DIFFO

- Anti-derivative = Indefinite integral
  = backwards diffo

eg $\int x^2 \, dx = \frac{x^3}{3} + c$

- Turns out anti-diffo gives same answer as working out area under curve.
- The anti-derivatives are all the curves with the same derivatives (ie same slopes along the curves)

③ DEFINITE INTEGRALS

$$\int_a^b f'(x) \, dx = \left[f(x)\right]_a^b = f(b) - f(a)$$

④ General Rule of Integration:

"Add one to power, divide by new power"

ie $\int x^p \, dx = \frac{x^{p+1}}{p+1} + c$ (p26)

eg $\int_0^2 x^2 \, dx = \left[\frac{x^3}{3}\right]_0^2 = \frac{8}{3} - 0 = \frac{8}{3}$

⑤ ALGEBRA INT

Tidy up algebra first (eg multiply or divide) then integrate.

⑥ LOGS INT

$\int \frac{1}{x} \, dx = \ln x + c$ (p26)

eg $\int \frac{1}{6x} \, dx = \frac{1}{6}\int \frac{1}{x} \, dx = \frac{1}{6}(\ln x + c)$

⑦ TRIG INT

$\int \cos x \, dx = \sin x + c$ (p26)

$\int \cos ax \, dx = \frac{1}{a}\sin ax + c$

$\int \sin x \, dx = -\cos x + c$ (p26)

$\int \sin ax \, dx = -\frac{1}{a}\cos ax + c$

- Angles must be in radians for integration.

eg $\int_0^{\frac{\pi}{2}} \sin 2x \, dx = \left[-\frac{1}{2}\cos 2x\right]_0^{\frac{\pi}{2}} = \frac{1}{2} + \frac{1}{2} = \boxed{1}$

- If you have to integrate a product, use prodsum formulas on p15

eg $\int \sin 4x \cos 2x \, dx = \frac{1}{2}\int 2\sin 4x \cos 2x \, dx$

$= \frac{1}{2}\int(\cos 6x + \cos 2x)\, dx$

etc.

⑧ EXP INT

(a) $\int e^x \, dx = e^x + c$ (p26)

$\int e^{ax} \, dx = \frac{1}{a} e^{ax} + c$

$\int e^{ax+b} \, dx = \frac{1}{a} e^{ax+b} + c$

eg $\int \left(e^{5x} - \frac{1}{e^{4x}}\right) dx = \int \left(e^{5x} - e^{-4x}\right) dx$

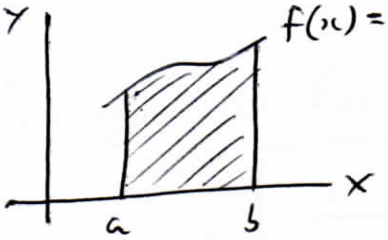
$= \frac{1}{5} e^{5x} + \frac{1}{4} e^{-4x} + c$

(b) $\int a^x \, dx = \frac{a^x}{\ln a} + c$ (p26)

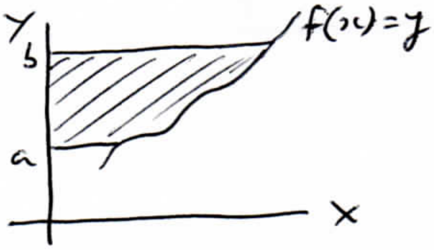
eg $\int 4^x \, dx = \frac{4^x}{\ln 4} + c$

9 AREA UNDER A CURVE

(a)



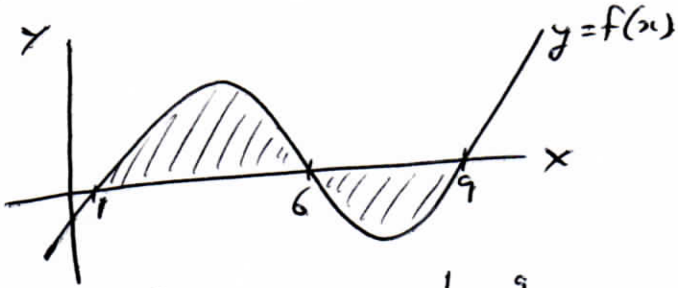
$$\text{Area} = \int_a^b y \, dx$$



$$\text{Area} = \int_a^b x \, dy$$

(b) If curve crosses the axis, do it in bits to make sure each bit is positive.

eg

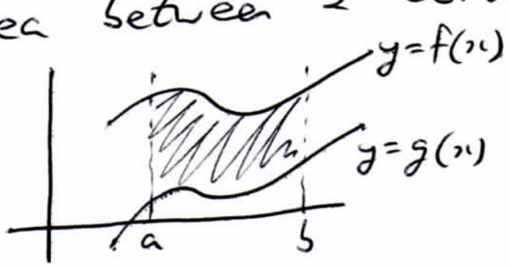


$$A = \int_1^6 f(x) \, dx + \left| \int_6^9 f(x) \, dx \right|$$

modulus makes sure area is positive.

(c) You may need to work out where curve crosses x-axis (ie $y = 0$)

(d) Area between 2 curves



$$A = \int_a^b (y_{high} - y_{low}) \, dx$$

You may need to find pts of intersection.

FORMULAE NOT IN TABLES BOOK

$$X^3 - Y^3 = (X - Y)(X^2 + XY + Y^2)$$

$$X^3 + Y^3 = (X + Y)(X^2 - XY + Y^2)$$

$x^2 - Sx + P = 0$ Quadratic equation from the roots

$x^3 - Sx^2 + SPPx - P = 0$ Cubic equation from the roots

Centroid of a triangle = $\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$

Period of y = sin θ or y = cos θ is 360^0 (or 2π)

P(A or B) = P(A) + P(B) – P(A and B)

P(A|B) = $\frac{P(A \cap B)}{P(B)}$

Independent events ......P(A|B) = P(A)

P(A and B) = P(A) x P(A|B)

Bernoulli is on p33 tables

Area under a curve (integration) = $\int y dx$

Margin of error (statistics) = $\frac{1}{\sqrt{n}}$

Arithmetic sequence: $T_n - T_{n-1} = d$

Geometric sequence: $\frac{T_n}{T_{n-1}} = r$

Complex numbers: $\overline{a + b\imath} = a - bi$

$$|a + bi| = \sqrt{a^2 + b^2}$$