Chapter 10 Exercise 10A

The answers given on page 261. No solutions necessary.

Exercise 10B

Q. 1. (i) $s_y = u \sin \alpha\, t - \frac{1}{2} g \cos \beta\, t^2 = 0$

$\Rightarrow t = 0$ **OR** $t = \frac{2u \sin \alpha}{g \cos \beta}$

$=$ time of flight

(ii) $v_y = 0 \Rightarrow u \sin \alpha - g \cos \beta\, t = 0$

$\Rightarrow t = \frac{u \sin \alpha}{g \cos \beta} = \frac{1}{2}$ (time of flight)

Q. 2. $v_x = u \cos 45° - gt \sin 30° = \frac{u}{\sqrt{2}} - \frac{gt}{2}$

$v_y = u \sin 45° - gt \cos 30° = \frac{u}{\sqrt{2}} - \frac{gt\sqrt{3}}{2}$

$s_x = ut \cos 45° - \frac{1}{2} gt^2 \sin 30° = \frac{ut}{\sqrt{2}} - \frac{gt^2}{4}$

$s_y = ut \sin 45° - \frac{1}{2} gt^2 \cos 30° = \frac{ut}{\sqrt{2}} - \frac{gt^2\sqrt{3}}{4}$

Range: s_x when $s_y = 0$

$\frac{ut}{\sqrt{2}} - \frac{gt^2\sqrt{3}}{4} = 0$ … multiply by $4\sqrt{2}$

$\Rightarrow 4ut - gt^2\sqrt{6} = 0$

$\Rightarrow t(4u - gt\sqrt{6}) = 0$

$\Rightarrow t = 0$ (Point of Projection) $\quad t = \frac{4u}{g\sqrt{6}}$ (Time of flight) … substitute into s_x

$\Rightarrow \text{Range} = \frac{u}{\sqrt{2}}\left[\frac{4u}{g\sqrt{6}}\right] - \frac{g}{4}\left[\frac{16u^2}{6g^2}\right]$

$\Rightarrow \text{Range} = \frac{4u^2}{g\sqrt{12}} - \frac{2u^2}{3g} = \frac{2u^2}{g\sqrt{3}} - \frac{2u^2}{3g}$

$= \frac{2u^2\sqrt{3} - 2u^2}{3g}$

$\Rightarrow \text{Range} = \frac{2u^2(\sqrt{3} - 1)}{3g}$

Maximum perpendicular height:
s_y when $v_y = 0$

$\frac{u}{\sqrt{2}} - \frac{gt\sqrt{3}}{2} = 0$ … multiply by 2

$\Rightarrow u\sqrt{2} - gt\sqrt{3} = 0$

$\Rightarrow t = \frac{u\sqrt{2}}{g\sqrt{3}}$ … substitute into s_y

$\Rightarrow$ Maximum perpendicular height

$= \frac{u}{\sqrt{2}}\left[\frac{u\sqrt{2}}{g\sqrt{3}}\right] - \frac{g\sqrt{3}}{4}\left[\frac{2u^2}{3g^2}\right]$

$= \frac{u^2}{g\sqrt{3}} - \frac{u^2}{2g\sqrt{3}}$

$= \frac{2u^2 - u^2}{2g\sqrt{3}}$

$= \frac{u^2}{2g\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$

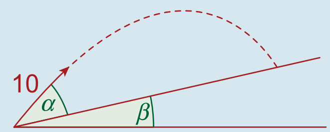
$= \frac{\sqrt{3}u^2}{6g}$

Q. 3. $\cos \alpha = \frac{4}{5}$

$\sin \alpha = \frac{3}{5}$

$\cos \beta = \frac{12}{13}$

$\sin \beta = \frac{5}{13}$



$s_y = 0$

$\Rightarrow 10 \sin \alpha\, t - \frac{1}{2} g \cos \beta\, t^2 = 0$

$\Rightarrow 6t - \frac{6}{13} gt^2 = 0$

$\Rightarrow t = 0$ **OR** $t = \frac{13}{g}$ = time of flight

At $t = \frac{13}{g}$, $s_x = 10 \cos \alpha\, t - \frac{1}{2} g \sin \beta\, t^2$

$= 10\left(\frac{4}{5}\right)\left(\frac{13}{g}\right) - \frac{1}{2} g\left(\frac{5}{13}\right)\left(\frac{169}{g^2}\right)$

$= \frac{143}{2g} = R$, the range

$\frac{2}{5}$ths of the time of flight $= \frac{2}{5} \times \frac{13}{g} = \frac{26}{5g}$

At $t = \frac{26}{5g}$, $s_x = 10\left(\frac{4}{5}\right)\left(\frac{26}{5g}\right) - \frac{1}{2} g\left(\frac{5}{13}\right)\left(\frac{676}{25g^2}\right)$

$= \frac{208}{5g} - \frac{26}{5g} = \frac{182}{5g}$

Now, $\frac{182}{5g} = \frac{28}{55}\left(\frac{143}{2g}\right) = \frac{28}{55} R$

Q. 4. $s_y = 10 \sin(\alpha - 30°)t - \frac{1}{2} \cos 30°\, gt^2$

$s_x = 10 \cos(\alpha - 30°)t - \frac{1}{2} \sin 30°\, gt^2$

(i) If $\alpha = 75°$,

$s_y = 10 \sin 45°\, t - \frac{1}{2} \cos 30°\, gt^2 = 0$

$\Rightarrow \frac{10}{\sqrt{2}} t - \frac{\sqrt{3}}{4} gt^2 = 0$

$\Rightarrow t = 0$ **OR** $t = \frac{40}{\sqrt{6}g}$

At $t = \frac{40}{\sqrt{6g}}$,

$$s_x = 10\left(\frac{1}{\sqrt{2}}\right)\left(\frac{40}{\sqrt{6g}}\right) - \frac{1}{2}\left(\frac{1}{2}\right)g\left(\frac{1{,}600}{6g}\right)$$

$$= \frac{200}{\sqrt{3g}} - \frac{200}{3g}$$

$$= \frac{200(\sqrt{3} - 1)}{3g}$$

(ii) If $\alpha = 60°$,

$$s_y = 10 \sin 30°\, t - \frac{1}{2}\cos 30°\, gt^2 = 0$$

$$\Rightarrow \frac{10}{\sqrt{2}}t - \frac{\sqrt{3}}{4}gt^2 = 0$$

$\Rightarrow t = 0$ **OR** $t = \frac{20}{\sqrt{3g}}$

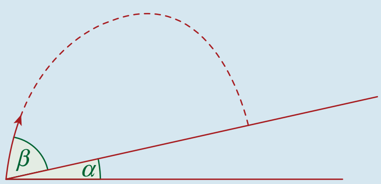
At $t = \frac{20}{\sqrt{3g}}$,

$$s_x = 10\left(\frac{\sqrt{3}}{2}\right)\left(\frac{20}{\sqrt{3g}}\right) - \frac{1}{2}\left(\frac{1}{2}\right)g\left(\frac{400}{3g}\right)$$

$$= \frac{10{,}000}{g^2} - \frac{100}{3g} = \frac{200}{3g}$$

Q. 5. $\tan^{-1} 2 = \cos^{-1}\frac{1}{\sqrt{5}} = \sin^{-1}\frac{2}{\sqrt{5}}$,

$$\vec{u} = 7\vec{i} + 14\vec{j}$$



Taking the horizontal as the x-axis.

$v_y = 0 \quad \Rightarrow \quad 14 - \frac{1}{2}gt = 0$

$\Rightarrow \quad t = \frac{28}{g}$ seconds

= time to reach greatest height above the horizontal.

Now taking the line of greatest slope as the x-axis, $\vec{u} = u\cos(\beta - \alpha)\vec{i} + u\sin(\beta - \alpha)\vec{j}$

But, $\cos(\beta - \alpha) = \cos\beta\cos\alpha + \sin\beta\sin\alpha$

$$= \frac{1}{\sqrt{5}}\cdot\frac{2}{\sqrt{5}} + \frac{2}{\sqrt{5}}\cdot\frac{1}{\sqrt{5}} = \frac{4}{5}$$

Therefore, $\sin(\beta - \alpha) = \frac{3}{5}$

$v_y = 0 \quad \Rightarrow \quad u\sin(\beta - \alpha) - \frac{1}{2}g\cos\alpha\, t = 0$

$$\Rightarrow 7\sqrt{5}\left(\frac{3}{5}\right) - \frac{1}{2}g\left(\frac{2}{\sqrt{5}}\right)t = 0$$

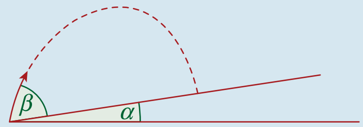
$$\Rightarrow \frac{21\sqrt{5}}{5} - \frac{\sqrt{5}}{5}gt = 0$$

$\Rightarrow t = \frac{21}{g}$ seconds

The ratio $t_1 : t_2 = \frac{28}{g} : \frac{21}{g} = 4 : 3$

Q. 6. $\sin\alpha = \frac{5}{13}$; $\cos\alpha = \frac{12}{13}$

$\sin\beta = \frac{4}{5}$; $\cos\beta = \frac{3}{5}$



$$s_x = u\cos\beta\, t - \frac{1}{2}g\sin\alpha\, t^2$$

$$= 10\left(\frac{3}{5}\right)t - \frac{1}{2}g\left(\frac{5}{13}\right)t^2$$

$$= 6t - \frac{5}{26}gt^2$$

$$s_y = u\sin\beta\, t - \frac{1}{2}g\cos\alpha\, t^2$$

$$= 10\left(\frac{4}{5}\right)t - \frac{1}{2}g\left(\frac{12}{13}\right)t^2$$

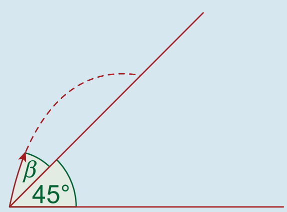
$$= 8t - \frac{6}{13}gt^2$$

$$s_x = 2s_y \Rightarrow 6t - \frac{5}{26}gt^2 = 2\left(8t - \frac{6}{13}gt^2\right)$$

$$\Rightarrow \frac{19}{26}gt^2 - 10t = 0$$

$\Rightarrow t = 0$ **OR** $t = \frac{260}{19g}$

Q. 7. $\sin\beta = \frac{1}{\sqrt{5}}$; $\cos\beta = \frac{2}{\sqrt{5}}$



$$s_y = 0 \Rightarrow u\sin\beta\, t - \frac{1}{2}g\cos 45°\, t^2 = 0$$

$$\Rightarrow 4t - \frac{1}{2\sqrt{2}}gt^2 = 0$$

$\Rightarrow t = 0$ **OR** $t = \frac{8\sqrt{2}}{g}$ = time of flight

$$v_x = u\cos\beta - g\sin 45°\, t = 8 - \frac{g}{\sqrt{2}}t$$

At $t = \frac{8\sqrt{2}}{g}$, $v_x = 8 - \frac{g}{\sqrt{2}}\left(\frac{8\sqrt{2}}{g}\right)$

$= 8 - 8 = 0$ m/s

$$v_y = u\sin\beta - g\cos 45°\, t = 4 - \frac{g}{\sqrt{2}}t$$

At $t = \frac{8\sqrt{2}}{g}$, $v_y = 4 - \frac{g}{\sqrt{2}}\left(\frac{8\sqrt{2}}{g}\right)$

$= 4 - 8 = -4$ m/s

$v = \sqrt{{v_x}^2 + {v_y}^2} = \sqrt{0 + 16} = 4$ m/s

Q. 8. (i) $s_y = 0 \Rightarrow \sqrt{26}\sin\alpha\, t - \frac{1}{2}g\left(\frac{5}{\sqrt{26}}\right)t^2 = 0$

$$\Rightarrow t = 0 \quad \textbf{OR} \quad t = \frac{52\sin\alpha}{5g}$$

(ii) $s_x = \sqrt{26}\cos\alpha\, t - \frac{1}{2}g\left(\frac{1}{\sqrt{26}}\right)t^2$

At $t = \frac{52\sin\alpha}{5g}$,

$$s_x = \sqrt{26}\cos\alpha\left(\frac{52\sin\alpha}{5g}\right) - \frac{1}{2}g\left(\frac{1}{\sqrt{26}}\right)\left(\frac{2{,}704\sin^2\alpha}{25g^2}\right)$$

$$= \frac{52\sqrt{26}\sin\alpha\cos\alpha}{5g} - \frac{52\sqrt{26}\sin^2\alpha}{25g}$$

$$= \frac{52\sqrt{26}\sin\alpha}{25g}(5\cos\alpha - \sin\alpha) = \text{the range}$$

(iii) If $\alpha = \beta$ then $\sin\alpha = \sin\beta = \frac{1}{\sqrt{26}}$; $\cos\alpha = \frac{5}{\sqrt{26}}$.

$$\therefore \text{The range} = \frac{52\sqrt{26}\left(\frac{1}{\sqrt{26}}\right)}{25g}\left(\frac{25}{\sqrt{26}} - \frac{1}{\sqrt{26}}\right)$$

$$= \frac{48\sqrt{26}}{25g}$$

(iv) If $\alpha = 2\beta$, then $\sin\alpha = \sin 2\beta$

$$= 2\sin\beta\cos\beta$$

$$= 2\left(\frac{1}{\sqrt{26}}\right)\left(\frac{5}{\sqrt{26}}\right)$$

$$= \frac{5}{13}$$

Also, $\cos\alpha = \cos 2\beta = \cos^2\beta - \sin^2\beta$

$$= \frac{25}{26} - \frac{1}{26}$$

$$= \frac{12}{13}$$

$$\therefore \text{The range} = \frac{52\sqrt{26}\left(\frac{5}{13}\right)}{25g}\left(\frac{60}{13} - \frac{5}{13}\right)$$

$$= \frac{44\sqrt{26}}{13g}$$

Q. 9. $v_x = u\cos\alpha + gt\sin\beta$

$v_y = u\sin\alpha - gt\cos\beta$

$s_x = ut\cos\alpha + \frac{1}{2}gt^2\sin\beta$

$s_y = ut\sin\alpha - \frac{1}{2}gt^2\cos\beta$

Range: s_x when $s_y = 0$

$s_y = 0$

$\Rightarrow ut\sin\alpha - \frac{1}{2}gt^2\cos\beta = 0$ … multiply by 2

$\Rightarrow 2ut\sin\alpha - gt^2\cos\beta = 0$

$\Rightarrow t(2u\sin\alpha - gt\cos\beta) = 0$

$\Rightarrow \underset{\text{Point of Projection}}{t = 0} \quad \underbrace{t = \frac{2u\sin\alpha}{g\cos\beta}}_{\text{Time of Flight}}$

Range $= s_x$ when $t = \dfrac{2u \sin \alpha}{g \cos \beta}$

$$\Rightarrow R = u\left[\frac{2u \sin \alpha}{g \cos \beta}\right]\cos \alpha + \frac{1}{2}g\left[\frac{4u^2 \sin^2\alpha}{g^2 \cos^2\beta}\right]\sin \beta$$

$$\Rightarrow R = \frac{2u^2 \sin \alpha \cos \alpha}{g \cos \beta} + \frac{2u^2 \sin^2 \alpha \sin \beta}{g \cos^2\beta}$$

$$\Rightarrow R = \frac{2u^2 \sin \alpha}{g \cos \beta}\left[\cos \alpha + \frac{\sin \alpha \sin \beta}{\cos \beta}\right]$$

$$\Rightarrow R = \frac{2u^2 \sin \alpha}{g \cos \beta}\left[\frac{\cos \alpha \cos \beta + \sin \alpha \sin \beta}{\cos \beta}\right]$$

$\ldots \cos A \cos B + \sin A \sin B = \cos(A - B)$

$$\Rightarrow R = \frac{2u^2}{g \cos^2 \beta}[\sin \alpha \cos(\alpha - \beta)]$$

$\ldots 2 \sin A \cos B = \sin(A + B) + \sin(A - B)$

$$\Rightarrow \sin A \cos B = \frac{1}{2}[\sin(A + B) + \sin(A - B)]$$

$$\Rightarrow R = \frac{2u^2}{g \cos^2 \beta}\left\{\frac{1}{2}[\sin(2\alpha - \beta) + \sin \beta]\right\}$$

$$\Rightarrow R = \frac{u^2}{g \cos^2 \beta}[\sin(2\alpha - \beta) + \sin \beta]$$

… as required

In this equation, u, g and β are fixed. Only α, the angle at which the projectile is fired, can vary.

The maximum value for the above expression is where $\sin(2\alpha - \beta) = 1$

$$\Rightarrow 2\alpha - \beta = 90°$$

$$\Rightarrow \alpha = \frac{\beta + 90°}{2}$$

Also, if $\sin(2\alpha - \beta) = 1$, then

$$R_{max} = \frac{u^2}{g \cos^2 \beta}[1 + \sin \beta]$$

$$\Rightarrow R_{max} = \frac{u^2}{g(1 - \sin^2 \beta)}[1 + \sin \beta]$$

$$\Rightarrow R_{max} = \frac{u^2}{g(1 - \sin \beta)\cancel{(1 + \sin \beta)}}\cancel{[1 + \sin \beta]}$$

$$\Rightarrow R_{max} = \frac{u^2}{g(1 - \sin \beta)}$$

(i) $\beta = 20° \Rightarrow \alpha = \dfrac{20° + 90°}{2} = 55°$

(ii) $\beta = 0° \Rightarrow \alpha = \dfrac{0° + 90°}{2} = 45°$

Q. 10. $v_x = u \cos \theta - gt \sin \alpha$

$v_y = u \sin \theta - gt \cos \alpha$

$s_x = ut \cos \theta - \frac{1}{2}gt^2 \sin \alpha$

$s_y = ut \sin \theta - \frac{1}{2}gt^2 \cos \alpha$

(i) For time of flight, let $s_y = 0$

$$\Rightarrow ut \sin \theta - \frac{1}{2}gt^2 \cos \alpha = 0$$

$$\Rightarrow 2ut \sin \theta - gt^2 \cos \alpha = 0$$

$$\Rightarrow t(2u \sin \theta - gt \cos \alpha) = 0$$

$$\Rightarrow \underset{\text{Point of Projection}}{t = 0} \qquad \underbrace{t = \frac{2u \sin \theta}{g \cos \alpha}}_{\text{Time of Flight}}$$

(ii) Range: s_x when $t = \dfrac{2u \sin \theta}{g \cos \alpha}$

$$\Rightarrow \text{Range} = u\left[\frac{2u \sin \theta}{g \cos \alpha}\right]\cos \theta - \frac{1}{2}g\left[\frac{4u^2 \sin^2\theta}{g^2 \cos^2\alpha}\right]\sin \alpha$$

$$\Rightarrow \text{Range} = \frac{2u^2 \sin \theta \cos \theta}{g \cos \alpha} - \frac{2u^2 \sin^2\theta \sin \alpha}{g \cos^2\alpha}$$

$$\Rightarrow \text{Range} = \frac{2u^2 \sin \theta \cos \theta \cos \alpha - 2u^2 \sin^2\theta \sin \alpha}{g \cos^2\alpha}$$

$$\Rightarrow \text{Range} = \frac{2u^2 \sin \theta[\cos \theta \cos \alpha - \sin \theta \sin \alpha]}{g \cos^2\alpha}$$

$$\Rightarrow \text{Range} = \frac{2u^2 \sin \theta[\cos(\theta + \alpha)]}{g \cos^2\alpha}$$

$$\Rightarrow \text{Range} = \frac{u^2[\sin(2\theta + \alpha) + \sin(-\alpha)]}{g\cos^2\alpha}$$

$$\Rightarrow \text{Range} = \frac{u^2[\sin(2\theta + \alpha) - \sin\alpha]}{g\cos^2\alpha}$$

Everything is fixed except θ. Therefore, maximum range occurs when $\sin(2\theta + \alpha) = 1$

$$\Rightarrow 2\theta + \alpha = \frac{\pi}{2}$$

$$\Rightarrow 2\theta = \frac{\pi}{2} - \alpha$$

$$\Rightarrow \theta = \frac{1}{2}\left(\frac{\pi}{2} - \alpha\right)$$

Exercise 10C

Q. 1. $s_y = 0 \Rightarrow u\sin(\alpha - \beta)t - \frac{1}{2}g\cos\beta\, t^2 = 0$

$$\Rightarrow t = 0 \quad \textbf{OR} \quad t = \frac{2u\sin(\alpha - \beta)}{g\cos\beta}$$

Now, $v_x = u\cos(\alpha - \beta) - g\sin\beta\, t$

But $v_x = 0$ at $t = \dfrac{2u\sin(\alpha - \beta)}{g\cos\beta}$

$$\therefore u\cos(\alpha - \beta) - g\sin\beta\left(\frac{2u\sin(\alpha - \beta)}{g\cos\beta}\right) = 0$$

Divide by $u\cos(\alpha - \beta)$.

$= 1 - 2\tan(\alpha - \beta)\tan\beta = 0$

$\Rightarrow 2\tan(\alpha - \beta)\tan\beta = 1$

If $(\alpha - \beta) = \frac{\pi}{4}$, then $\tan(\alpha - \beta) = 1$

and hence $2\tan\beta = 1$

$$\Rightarrow \tan\beta = \frac{1}{2}$$

$$\Rightarrow \sin\beta = \frac{1}{\sqrt{5}} \text{ and } \cos\beta = \frac{2}{\sqrt{5}}$$

In this case $t = \dfrac{2u\sin(\alpha - \beta)}{g\cos\beta}$

$$\Rightarrow t = \frac{2u\left(\frac{1}{\sqrt{2}}\right)}{g(2/\sqrt{5})} = \frac{\sqrt{5}u}{\sqrt{2}g}$$

$s_x = u\cos(\alpha - \beta)t - \frac{1}{3}g\sin\beta\, t^2.$

At $t = \dfrac{\sqrt{5}u}{\sqrt{2}g}$,

$$s_x = u\left(\frac{1}{\sqrt{2}}\right)\left(\frac{\sqrt{5}u}{\sqrt{2}g}\right) - \frac{1}{2}g\left(\frac{1}{\sqrt{5}}\right)\left(\frac{5u^2}{2g^2}\right)$$

$$= \frac{\sqrt{5}u^2}{2g} - \frac{\sqrt{5}u^2}{4g}$$

$$= \frac{\sqrt{5}u^2}{4g} = \text{Range}$$

Q. 2. $v_x = u\cos 30° - gt\sin 30° = \dfrac{u\sqrt{3}}{2} - \dfrac{gt}{2}$

$v_y = u\cos 30° - gt\cos 30° = \dfrac{u}{2} - \dfrac{gt\sqrt{3}}{2}$

$s_x = ut\cos 30° - \frac{1}{2}gt^2\sin 30° = \dfrac{ut\sqrt{3}}{2} - \dfrac{gt^2}{4}$

$s_y = ut\sin 30° - \frac{1}{2}gt^2\cos 30° = \dfrac{ut}{2} - \dfrac{gt^2\sqrt{3}}{4}$

For landing angle, need v_x and v_y when

$s_y = 0$

$\dfrac{ut}{2} - \dfrac{gt^2\sqrt{3}}{4} = 0$ … multiply by 4

$\Rightarrow 2ut - gt^2\sqrt{3} = 0$

$\Rightarrow t(2u - gt\sqrt{3}) = 0$

$\Rightarrow t = 0$ (Point of Projection) $\quad t = \dfrac{2u}{g\sqrt{3}}$ (Time of Flight)

$$v_x = \frac{u\sqrt{3}}{2} - \frac{g}{2}\left[\frac{2u}{g\sqrt{3}}\right] = \frac{u\sqrt{3}}{2} - \frac{u}{\sqrt{3}}$$

$$= \frac{3u - 2u}{2\sqrt{3}}$$

$$= \frac{u}{2\sqrt{3}}$$

$$v_y = \frac{u}{2} - \frac{g\sqrt{3}}{2}\left[\frac{2u}{g\sqrt{3}}\right] = \frac{u}{2} - u = -\frac{u}{2}$$

Let landing angle $= \theta$

$$\tan\theta = -\frac{v_y}{v_x} = \frac{u}{2} \times \frac{2\sqrt{3}}{u} = \sqrt{3}$$

$$\Rightarrow \theta = \tan^{-1}\sqrt{3} = 60°$$

Q. 3. $s_y = 0 \Rightarrow u \sin(\alpha - 45°)t - \frac{1}{2}g \cos 45°\, t^2 = 0$

$\Rightarrow u \sin(\alpha - 45°)t - \frac{1}{2\sqrt{2}} g\, t^2 = 0$

$\Rightarrow t = 0$ **OR** $t = \frac{2\sqrt{2} u \sin(\alpha - 45°)}{g}$

$= \text{time of flight}$

$v_y = u \sin(\alpha - 45°) - \frac{g}{\sqrt{2}} t$

At time of flight,

$v_y = u \sin(\alpha - 45°) - \frac{g}{2}\left(\frac{2\sqrt{2}\, u \sin(\alpha - 45°)}{g}\right)$

$= -u \sin(\alpha - 45°)$

$v_x = u \cos(\alpha - 45°) - \frac{g}{\sqrt{2}} t$

At time of flight,

$v_x = u \cos(\alpha - 45°) - \frac{g}{\sqrt{2}}\left(\frac{2\sqrt{2}\, u \sin(\alpha - 45°)}{g}\right)$

$= u \cos(\alpha - 45°) - 2u \sin(\alpha - 45°)$

$\tan \beta = \frac{-v_y}{v_x}$

$= \frac{u \sin(\alpha - 45°)}{u \cos(\alpha - 45°) - 2u \sin(\alpha - 45°)}$

$= \frac{\tan(\alpha - 45°)}{1 - 2 \tan(\alpha - 45°)}$

But $\tan(\alpha - 45°) = \frac{\tan \alpha - \tan 45°}{1 + \tan \alpha \tan 45°}$

$= \frac{\tan \alpha - 1}{1 + \tan \alpha}$

$\therefore \tan \beta = \frac{\frac{\tan \alpha - 1}{1 + \tan \alpha}}{1 - 2\left(\frac{\tan \alpha - 1}{1 + \tan \alpha}\right)}$

$= \frac{\tan \alpha - 1}{1 + \tan \alpha - 2 \tan \alpha + 2}$

$= \frac{\tan \alpha - 1}{3 - \tan \alpha}$

(i) If it lands horizontally, then $\beta = 45°$

$\Rightarrow 1 = \frac{\tan \alpha - 1}{3 - \tan \alpha}$

$\Rightarrow 3 - \tan \alpha = \tan \alpha - 1$

$\Rightarrow \tan \alpha = 2$

(ii) If it lands vertically, then $\beta = 90°$

$\Rightarrow \frac{\tan \alpha - 1}{3 - \tan \alpha} = \text{undefined}$

$\therefore 3 - \tan \alpha = 0$

$\therefore \tan \alpha = 3$

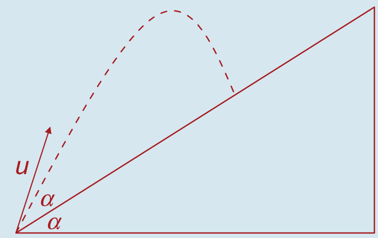
(iii) $\frac{\tan \alpha - 1}{3 - \tan \alpha} = \frac{1}{3}$

$\therefore 3 \tan \alpha - 3 = 3 - \tan \alpha$

$\therefore 4 \tan \alpha = 6$

$\therefore \tan \alpha = \frac{6}{4} = \frac{3}{2}$

Q. 4.



$\tan \alpha = \frac{1}{\sqrt{2}} \Rightarrow \cos \alpha = \frac{\sqrt{2}}{\sqrt{3}}$ and $\sin a = \frac{1}{\sqrt{3}}$

$v_x = u \cos \alpha - gt \sin \alpha = \frac{u\sqrt{2}}{\sqrt{3}} - \frac{gt}{\sqrt{3}}$

$v_y = u \sin \alpha - gt \cos \alpha = \frac{u}{\sqrt{3}} - \frac{gt\sqrt{2}}{\sqrt{3}}$

$s_x = ut \cos \alpha - \frac{1}{2}gt^2 \sin \alpha = \frac{ut\sqrt{2}}{\sqrt{3}} - \frac{gt^2}{2\sqrt{3}}$

$s_y = ut \sin \alpha - \frac{1}{2}gt^2 \cos \alpha = \frac{ut}{\sqrt{3}} - \frac{gt^2\sqrt{2}}{2\sqrt{3}}$

(i) Lands at right angles if $v_x = 0$ when

$s_y = 0$

$\Rightarrow \frac{ut}{\sqrt{3}} - \frac{gt^2\sqrt{2}}{2\sqrt{3}} = 0$

$\Rightarrow 2ut - gt^2\sqrt{2} = 0$

$\Rightarrow t(2u - gt\sqrt{2}) = 0$

$\Rightarrow \underbrace{t = 0}_{\text{Point of Projection}} \qquad \underbrace{t = \frac{2u}{g\sqrt{2}} = \frac{u\sqrt{2}}{g}}_{\text{Time of Flight}}$

$v_x = \frac{u\sqrt{2}}{\sqrt{3}} - \frac{g}{\sqrt{3}}\left[\frac{u\sqrt{2}}{g}\right]$

$\Rightarrow v_x = \frac{u\sqrt{2}}{\sqrt{3}} - \frac{u\sqrt{2}}{\sqrt{3}} = 0$

$\Rightarrow$ Strikes plane at right angles

(ii) Range: Find s_x when $t = \frac{u\sqrt{2}}{g}$

$\Rightarrow \text{Range} = \frac{u\sqrt{2}}{\sqrt{3}}\left[\frac{u\sqrt{2}}{g}\right] - \frac{g}{2\sqrt{3}}\left[\frac{2u^2}{g^2}\right]$

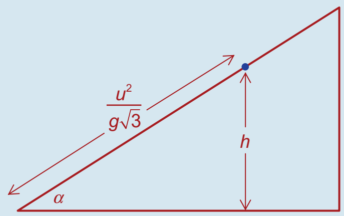
$\Rightarrow \text{Range} = \frac{2u^2}{g\sqrt{3}} - \frac{u^2}{g\sqrt{3}} = \frac{u^2}{g\sqrt{3}}$

(iii) Total Energy in the beginning $= \frac{1}{2}mu^2$

Total Energy at point of landing

$= \frac{1}{2}mv^2 + mgh$

… where h = height above take-off position



$$\sin\alpha = \frac{h}{\frac{u^2}{g\sqrt{3}}} = h\left(\frac{g\sqrt{3}}{u^2}\right)$$

$$\Rightarrow \frac{1}{\sqrt{3}} = h\left(\frac{g\sqrt{3}}{u^2}\right)$$

$$\Rightarrow 3gh = u^2$$

$$\Rightarrow h = \frac{u^2}{3g}$$

Velocity at landing is v_y when $t = \frac{u\sqrt{2}}{g}$

since $v_x = 0$ at landing

$$\Rightarrow v = \frac{u}{\sqrt{3}} - \frac{g\sqrt{2}}{\sqrt{3}}\left[\frac{u\sqrt{2}}{g}\right] = \frac{u}{\sqrt{3}} - \frac{2u}{\sqrt{3}}$$

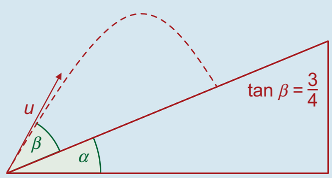
$$= -\frac{u}{\sqrt{3}}$$

$\Rightarrow$ Total Energy at landing

$$= \underbrace{\frac{1}{2}m\left[-\frac{u}{\sqrt{3}}\right]^2}_{\text{Kinetic Energy}} + \underbrace{mg\left[\frac{u^2}{3g}\right]}_{\text{Potential Energy}}$$

$$= \frac{mu^2}{6} + \frac{mu^2}{3} = \frac{1}{2}mu^2$$

Q. 5.



$\tan\alpha = \frac{3}{4}$, $\tan\beta = \frac{1}{2}$,

To prove: $\tan\theta = 2$

$$u_x = u\cos\beta$$

$$u_y = u\sin\beta$$

$$a_x = -g\sin\alpha$$

$$a_y = -g\cos\alpha$$

$$\therefore v_x = u\cos\beta - g\sin\alpha\, t$$

$$v_y = u\sin\beta - g\cos\alpha\, t$$

$$s_x = u\cos\beta\, t - \frac{1}{2}g\sin\alpha\, t^2$$

$$s_y = u\sin\beta\, t - \frac{1}{2}g\cos\alpha\, t^2$$

It lands when $s_y = 0$

$$\Rightarrow u\sin\beta\, t - \frac{1}{2}g\cos\alpha\, t^2 = 0$$

$$\Rightarrow t = 0 \quad \textbf{OR} \quad t = \frac{2u\sin\beta}{g\cos\alpha} = \text{time of flight}$$

At this time, $v_x = u\cos\beta - g\sin\alpha\left(\frac{2u\sin\beta}{g\cos\alpha}\right)$

$$= u\cos\beta - 2u\tan\alpha\sin\beta$$

$$v_y = u\sin\beta - g\cos\alpha\left(\frac{2u\sin\beta}{g\cos\alpha}\right) = -u\sin\beta$$

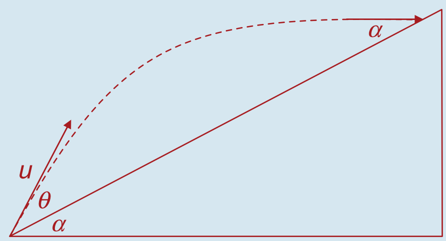
$$\tan\theta = -\frac{v_y}{v_x}$$

$$= \frac{u\sin\beta}{u\cos\beta - 2u\tan\alpha\sin\beta} \quad (\div u\cos\beta)$$

$$= \frac{\tan\beta}{1 - 2\tan\alpha\tan\beta}$$

$$= \frac{\frac{1}{2}}{1 - 2\left(\frac{3}{4}\right)\left(\frac{1}{2}\right)} = \frac{\frac{1}{2}}{1 - \frac{3}{4}} = 2 \quad \text{QED.}$$

Q. 6.



$\tan\alpha = \frac{1}{2} \Rightarrow \cos\alpha = \frac{2}{\sqrt{5}}$ and $\sin\alpha = \frac{1}{\sqrt{5}}$

$$v_x = u\cos\theta - gt\sin\alpha$$

$$= u\cos\theta - \frac{gt}{\sqrt{5}}$$

$$v_y = u\sin\theta - gt\cos\alpha$$

$$= u\sin\theta - \frac{2gt}{\sqrt{5}}$$

$$s_x = ut\cos\theta - \frac{1}{2}gt^2\sin\alpha$$

$$= ut\cos\theta - \frac{gt^2}{2\sqrt{5}}$$

$$s_y = ut\sin\theta - \frac{1}{2}gt^2\cos\alpha$$

$$= ut\sin\theta - \frac{gt^2}{\sqrt{5}}$$

For time of flight, let $s_y = 0$

$$\Rightarrow ut\sin\theta - \frac{gt^2}{\sqrt{5}} = 0 \quad \text{... multiply by } \sqrt{5}$$

$$ut\sqrt{5}\sin\theta - gt^2 = 0$$

$$t(u\sqrt{5}\sin\theta - gt) = 0$$

$$\Rightarrow \underbrace{t = 0}_{\text{Point of Projection}} \qquad \underbrace{t = \frac{u\sqrt{5}\sin\theta}{g}}_{\text{Time of Flight}}$$

For landing angle, find v_x and v_y when $t = \frac{u\sqrt{5}\sin\theta}{g}$

$$v_x = u\cos\theta - \frac{g}{\sqrt{5}}\left[\frac{u\sqrt{5}\sin\theta}{g}\right]$$

$$= u(\cos\theta - \sin\theta)$$

$$v_y = u\sin\theta - \frac{2g}{\sqrt{5}}\left[\frac{u\sqrt{5}\sin\theta}{g}\right]$$

$$= u\sin\theta - 2u\sin\theta = -u\sin\theta$$

We know that the landing angle is α where $\tan\alpha = \frac{1}{2}$

We also know that $\tan\alpha = \frac{-v_y}{v_x}$

$$\Rightarrow \frac{-v_y}{v_x} = \frac{1}{2}$$

$$\Rightarrow \frac{u\sin\theta}{u(\cos\theta - \sin\theta)} = \frac{1}{2}$$

$\Rightarrow 2\sin\theta = \cos\theta - \sin\theta$

$\Rightarrow 3\sin\theta = \cos\theta$ ... divide by $\cos\theta$

$\Rightarrow 3\tan\theta = 1$

$\Rightarrow \tan\theta = \frac{1}{3}$ ... as required

$\Rightarrow$ Range: Find s_x when $t = \frac{u\sqrt{5}\sin\theta}{g}$

$$\Rightarrow \text{Range} = ut\cos\theta - \frac{gt^2}{2\sqrt{5}}$$

... $\tan\theta = \frac{1}{3} \Rightarrow \cos\theta = \frac{3}{\sqrt{10}}$ and $\sin\theta = \frac{1}{\sqrt{10}}$

$\Rightarrow$ Range

$$= u\left[\frac{u\sqrt{5}}{g}\left(\frac{1}{\sqrt{10}}\right)\right]\left[\frac{3}{\sqrt{10}}\right] - \frac{g}{2\sqrt{5}}\left[\frac{5u^2}{g^2}\left(\frac{1}{10}\right)\right]$$

$$\Rightarrow \text{Range} = \frac{3u^2\sqrt{5}}{10g} - \frac{u^2}{4g\sqrt{5}}$$

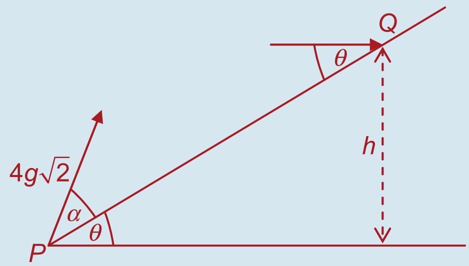
$$\Rightarrow \text{Range} = \frac{30u^2 - 5u^2}{20g\sqrt{5}}$$

$$= \frac{25u^2}{20g\sqrt{5}}$$

$$= \frac{5u^2}{4g\sqrt{5}}$$

$$= \frac{u^2\sqrt{5}}{4g}$$

Q. 7.



$\tan\alpha = \frac{1}{3} \Rightarrow \cos\alpha = \frac{3}{\sqrt{10}}$ and $\sin\alpha = \frac{1}{\sqrt{10}}$

(a) $v_x = u\cos\alpha - gt\sin\theta$

$$= 4g\sqrt{2}\left(\frac{3}{\sqrt{10}}\right) - gt\sin\theta$$

$$= \frac{12g}{\sqrt{5}} - gt\sin\theta$$

$$v_y = u\sin\alpha - gt\cos\theta$$

$$= 4g\sqrt{2}\left(\frac{1}{\sqrt{10}}\right) - gt\cos\theta$$

$$= \frac{4g}{\sqrt{5}} - gt\cos\theta$$

$$s_x = ut\cos\alpha - \frac{1}{2}gt^2\sin\theta$$

$$= 4g\sqrt{2}t\left(\frac{3}{\sqrt{10}}\right) - \frac{1}{2}gt^2\sin\theta$$

$$= \frac{12gt}{\sqrt{5}} - \frac{1}{2}gt^2\sin\theta$$

$$s_y = ut\sin\alpha - \frac{1}{2}gt^2\cos\theta$$

$$= 4g\sqrt{2}\left(\frac{1}{\sqrt{10}}\right) - \frac{1}{2}gt^2\cos\theta$$

$$= \frac{4gt}{\sqrt{5}} - \frac{1}{2}gt^2\cos\theta$$

For landing angle, need v_x and v_y when $s_y = 0$

$\frac{4gt}{\sqrt{5}} - \frac{1}{2}gt^2\cos\theta = 0$ ... multiply by $2\sqrt{5}$

$\Rightarrow 8gt - gt^2\sqrt{5}\cos\theta = 0$ ... divide by g

$\Rightarrow 8t - t^2\sqrt{5}\cos\theta = 0$

$t(8 - t\sqrt{5}\cos\theta) = 0$

$$\Rightarrow \underbrace{t = 0}_{\text{Point of Projection}} \qquad \underbrace{t = \frac{8}{\sqrt{5}\cos\theta}}_{\text{Time of Flight}}$$

$$v_x = \frac{12g}{\sqrt{5}} - g\left[\frac{8}{\sqrt{5}\cos\theta}\right]\sin\theta$$

$$= \frac{12g}{\sqrt{5}} - \frac{8g\tan\theta}{\sqrt{5}}$$

$$= \frac{4g(3 - 2\tan\theta)}{\sqrt{5}}$$

$$v_y = \frac{4g}{\sqrt{5}} - g\left[\frac{8}{\sqrt{5}\cos\theta}\right]\cos\theta$$

$$= \frac{4g}{\sqrt{5}} - \frac{8g}{\sqrt{5}}$$

$$= -\frac{4g}{\sqrt{5}}$$

$$\tan\theta = \frac{v_y}{v_x}$$

$$= \frac{4g}{\sqrt{5}} \times \frac{\sqrt{5}}{4g(3 - 2\tan\theta)}$$

$$\Rightarrow \tan\theta = \frac{1}{3 - 2\tan\theta}$$

$$\Rightarrow 3\tan\theta - 2\tan^2\theta = 1$$

$$\Rightarrow 2\tan^2\theta - 3\tan\theta + 1 = 0$$

$$\Rightarrow (2\tan\theta - 1)(\tan\theta - 1) = 0$$

$$\Rightarrow \tan\theta = \frac{1}{2} \quad \textbf{OR} \quad \tan\theta = 1$$

$$\Rightarrow \theta = \tan^{-1}\left(\frac{1}{2}\right) \quad \textbf{OR} \quad \theta = 45^\circ$$

(b) (i) Let $\tan\theta = 0.5$

$$\Rightarrow v_x = \frac{8g}{\sqrt{5}} \quad \text{and} \quad v_y = -\frac{4g}{\sqrt{5}}$$

$$\Rightarrow |v| = \sqrt{\left(\frac{8g}{\sqrt{5}}\right)^2 + -\left(\frac{4g}{\sqrt{5}}\right)^2}$$

$$= \sqrt{\frac{64g^2}{5} + \frac{16g^2}{5}}$$

$$= \sqrt{\frac{80g^2}{5}}$$

$$= \sqrt{16g^2} = 4g \text{ m/s}$$

(ii) Total Energy at $P = \frac{1}{2}mu^2$

$$= \frac{1}{2}m(4g\sqrt{2})^2$$

$$= \frac{1}{2}m(32g^2)$$

$$= 16mg^2$$

Total Energy at $Q = \frac{1}{2}mv^2 + mgh$

In order to find h, must firstly find range.

i.e. find s_x when $t = \frac{8}{\sqrt{5}\cos\theta}$

$$= \frac{8}{\sqrt{5}\left(\frac{2}{\sqrt{5}}\right)} = 4$$

$$\text{Range} = \frac{12g(4)}{\sqrt{5}} - \frac{1}{2}g(16)\left(\frac{1}{\sqrt{5}}\right)$$

$$= \frac{48g}{\sqrt{5}} - \frac{8g}{\sqrt{5}}$$

$$= \frac{40g}{\sqrt{5}}$$

$$= 8g\sqrt{5}$$

$$\sin\theta = \frac{h}{\text{Range}}$$

$$\Rightarrow \frac{1}{\sqrt{5}} = \frac{h}{8g\sqrt{5}}$$

$$\Rightarrow h = 8g$$

$\Rightarrow$ Total Energy at Q

$$= \frac{1}{2}m(4g)^2 + mg(8g)$$

$$= 8mg^2 + 8mg^2$$

$$= 16mg^2$$

This is the same as the total energy at P.

Q. 8. $v_x = u\cos\theta - gt\sin\alpha$

$$v_y = u\cos\theta - gt\cos\alpha$$

$$s_x = ut\cos\theta - \frac{1}{2}gt^2\sin\alpha$$

$$s_y = ut\sin\theta - \frac{1}{2}gt^2\cos\alpha$$

For landing angle, need v_x and v_y when $s_y = 0$

$$ut\sin\theta - \frac{1}{2}gt^2\cos\alpha = 0$$

$$\Rightarrow 2ut\sin\theta - gt^2\cos\alpha = 0$$

$$\Rightarrow t(2u\sin\theta - gt\cos\alpha) = 0$$

$$\Rightarrow t = 0 \qquad t = \frac{2u\sin\theta}{g\cos\alpha}$$

Point of Projection — Time of Flight

$$v_x = u\cos\theta - g\left[\frac{2u\sin\theta}{g\cos\alpha}\right]\sin\alpha$$

$$= u\cos\theta - 2u\sin\theta\tan\alpha$$

$$= u(\cos\theta - 2\sin\theta\tan\alpha)$$

$$v_y = u\sin\theta - g\left[\frac{2u\sin\theta}{g\cos\alpha}\right]\cos\alpha$$

$$= u\sin\theta - 2u\sin\theta$$

$$= -u\sin\theta$$

Lands horizontally

$\Rightarrow$ Landing angle $= \alpha$

$$\tan\alpha = \frac{-v_y}{v_x}$$

$$= \frac{u\sin\theta}{u(\cos\theta - 2\sin\theta\tan\alpha)}$$

$$\Rightarrow \tan\alpha = \frac{\sin\theta}{\cos\theta - 2\sin\theta\tan\alpha}$$

$$\Rightarrow \sin\theta = \tan\alpha\cos\theta - 2\sin\theta\tan^2\alpha$$

... divide by $\cos\theta$

$$\Rightarrow \tan\theta = \tan\alpha - 2\tan\theta\tan^2\alpha$$

$$\Rightarrow \tan\theta(1 + 2\tan^2\alpha) = \tan\alpha$$

$$\Rightarrow \tan\theta = \frac{\tan\alpha}{1 + 2\tan^2\alpha}$$

$$\Rightarrow \tan\theta = \frac{\frac{\sin\alpha}{\cos\alpha}}{1 + \frac{2\sin^2\alpha}{\cos^2\alpha}}$$

... multiply top and bottom by $\cos^2\alpha$

$$\Rightarrow \tan\theta = \frac{\sin\alpha\cos\alpha}{\cos^2\alpha + 2\sin^2\alpha}$$

$$\Rightarrow \tan\theta = \frac{\sin\alpha\cos\alpha}{\cos^2\alpha + 2(1 - \cos^2\alpha)}$$

$$\Rightarrow \tan\theta = \frac{\sin\alpha\cos\alpha}{\cos^2\alpha + 2 - 2\cos^2\alpha}$$

$$\Rightarrow \tan\theta = \frac{\sin\alpha\cos\alpha}{2 - \cos^2\alpha}$$

Q. 9.

$$v_x = u\cos\theta - gt\sin 60^\circ$$

$$= u\cos\theta - \frac{gt\sqrt{3}}{2}$$

$$v_y = u\sin\theta - gt\cos 60^\circ$$

$$= u\sin(\theta - 60^\circ) - \frac{gt}{2}$$

Note: θ is the angle between the projection and the plane.

$$s_x = ut\cos\theta - \frac{1}{2}gt^2\sin 60^\circ$$

$$= ut\cos\theta - \frac{gt^2\sqrt{3}}{4}$$

$$s_y = ut\sin\theta - \frac{1}{2}gt^2\cos 60^\circ$$

$$= ut\sin\theta - \frac{gt^2}{4}$$

Maximum Perpendicular Height: Find s_y when $v_y = 0$

$$u\sin\theta - \frac{gt}{2} = 0$$

$$\Rightarrow 2u\sin\theta - gt = 0$$

$$\Rightarrow t = \frac{2u\sin\theta}{g}$$

$$\Rightarrow H = u\left[\frac{2u\sin\theta}{g}\right]\sin\theta - \frac{g}{4}\left[\frac{4u^2\sin^2\theta}{g^2}\right]$$

$$\Rightarrow H = \frac{2u^2\sin^2\theta}{g} - \frac{u^2\sin^2\theta}{g}$$

$$\Rightarrow H = \frac{u^2\sin^2\theta}{g}$$

For the second part of the question, we let $s_y = H\sin^2\theta$

$$\Rightarrow ut\sin\theta - \frac{gt^2}{4}$$

$$= \left[\frac{u^2\sin^2\theta}{g}\right]\sin^2\theta \quad \text{... multiply by } 4g$$

$$\Rightarrow 4gut\sin\theta - g^2t^2 = 4u^2\sin^4\theta$$

$$\Rightarrow g^2t^2 - 4gut\sin\theta + 4u^2\sin^4\theta = 0$$

This is a quadratic equation in t

$$\Rightarrow t = \frac{4gu\sin\theta \pm \sqrt{16g^2u^2\sin^2\theta - 16g^2u^2\sin^4\theta}}{2g^2}$$

$$\Rightarrow t = \frac{4gu\sin\theta \pm \sqrt{16g^2u^2\sin^2\theta(1 - \sin^2\theta)}}{2g^2}$$

$$\Rightarrow t = \frac{4gu\sin\theta \pm \sqrt{16g^2u^2\sin^2\theta\cos^2\theta}}{2g^2}$$

$$\Rightarrow t = \frac{4gu\sin\theta \pm 4gu\sin\theta\cos\theta}{2g^2}$$

... divide top and bottom by $2g$

$$\Rightarrow t = \frac{2u\sin\theta \pm 2u\sin\theta\cos\theta}{g}$$

$$\Rightarrow t_1 = \frac{2u\sin\theta + 2u\sin\theta\cos\theta}{g}$$

$$= \frac{2u\sin\theta(1 + \cos\theta)}{g}$$

$$\text{and } t_2 = \frac{2u\sin\theta - 2u\sin\theta\cos\theta}{g}$$

$$= \frac{2u\sin\theta(1 - \cos\theta)}{g}$$

$$t_1 - t_2 = \frac{2u\sin\theta(1 + \cos\theta)}{g} - \frac{2u\sin\theta(1 - \cos\theta)}{g}$$

$$= \frac{2u\sin\theta}{g}[1 + \cos\theta - (1 - \cos\theta)]$$

$$= \frac{2u\sin\theta}{g}(2\cos\theta)$$

$$= \frac{2u[2\sin\theta\cos\theta]}{g}$$

... $2\sin\theta\cos\theta = \sin 2\theta$

$$= \frac{2u\sin 2\theta}{g}$$

Exercise 10D

Q. 1. (i) **Speed at impact:**

$v^2 = u^2 + 2as$

$\Rightarrow v^2 = 0^2 + 2gh$

$\Rightarrow v = \sqrt{2gh}$

Speed after impact $= \frac{3}{4}\sqrt{2gh}$

New Height:

$v^2 = u^2 + 2as$

$\Rightarrow 0^2 = \frac{9}{16}(2gh) + 2(-g)s$

$\Rightarrow s = \frac{9}{16}h$

(ii) Each new height $= \frac{9}{16}$ths of the previous height

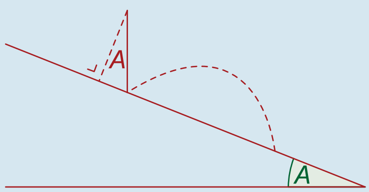
$\therefore$ Total distance $= 2h + \frac{9}{16}(2h) + \left(\frac{9}{16}\right)^2(2h) + \dots$

$= s_\infty$ of a G.P. with

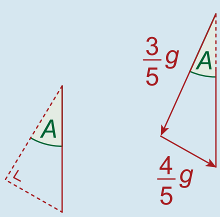
$a = 2h,\ r = \frac{9}{16}$

$= \dfrac{2h}{1 - \frac{9}{16}} = \dfrac{32h}{7}$

Q. 2. (i)



Gravity **Resolved**



$\vec{u} = u \sin A\vec{i} - u \cos A\vec{j}$

$= 20\left(\frac{4}{5}\right)\vec{i} - 20\left(\frac{3}{5}\right)\vec{j} = 16\vec{i} - 12\vec{j}$

Before	**(Mass)**	**After**
$16\vec{i} - 12\vec{j}$	M	$16\vec{i} + p\vec{j}$

$\frac{\text{NEW}}{\text{OLD}} = -e$

$\Rightarrow \frac{p}{-12} = \frac{-2}{3}$

$\Rightarrow p = 8$

New initial speed $= 16\vec{i} + 8\vec{j}$

Find length of hop means find s_x when $s_y = 0$

$s_y = 0$

$\Rightarrow 8t - \frac{1}{2}\left(\frac{3}{5}g\right)t^2 = 0$

$\Rightarrow t = 0$ **OR** $t = \frac{80}{3g}$

At $t = \frac{80}{3g}$,

$s_x = 16t + \frac{1}{2}\left(\frac{4}{5}g\right)t^2$

$= 16\left(\frac{80}{3g}\right) + \frac{1}{2}\left(\frac{4}{5}g\right)\left(\frac{6,400}{9g^2}\right)$

$= \frac{6,400}{9g}$ metres

(ii) At $t = \frac{80}{3g}$,

$v_x = 16 + \frac{4}{5}gt$

$= 16 + \frac{4}{5}g\left(\frac{80}{3g}\right) = \frac{112}{3}$

At $t = \frac{80}{3g}$,

$v_y = 8 - \frac{3}{5}gt$

$= 8 - \frac{3}{5}g\left(\frac{80}{3g}\right)$

$= -8$

If L is the landing angle, $\tan L = \dfrac{-v_y}{v_x}$

$= \dfrac{8}{\frac{112}{3}}$

$= \frac{3}{14}$

$\Rightarrow \tan L = 0.2143$

$\Rightarrow L = 12°\ 6' = 12°$ (to nearest degree)

Q. 3. **Before** **After**

$u \sin A\vec{i} - u \cos A\vec{j}$ $u \sin A\vec{i} + p\vec{j}$

$\frac{\text{NEW}}{\text{OLD}} = -e$

$\Rightarrow \frac{p}{-u \cos A} = \frac{-1}{4}$

$\Rightarrow p = \frac{1}{4}u \cos A$

Initial Speed $= u \sin A\vec{i} + \frac{1}{4} u \cos A\vec{j}$

$s_y = 0$

$\Rightarrow \frac{1}{4} u \cos At - \frac{1}{2}g \cos At^2 = 0$

$\Rightarrow t = 0$ **OR** $t = \frac{u}{2g}$

At $t = \frac{u}{2g}$,

$s_x = u \sin At + \frac{1}{2}g \sin At^2$

$= u \sin A\left(\frac{u}{2g}\right) + \frac{1}{2}g \sin A\left(\frac{u^2}{4g^2}\right)$

$= \frac{5u^2 \sin A}{8g}$

At the first hop $\left(\text{when } t = \frac{u}{2g}\right)$,

$v_x = u \sin A + g \sin At$

$= u \sin A + g \sin A\left(\frac{u}{2g}\right)$

$= \frac{3}{2}u \sin A$

$v_y = \frac{1}{4}u \cos A - g \cos At$

$= \frac{1}{4}u \cos A - g \cos A\left(\frac{u}{2g}\right)$

$= -\frac{1}{4}u \cos A$

After the first hop,

$v_x = \frac{3}{2}u \sin A$ and $v_y = -\frac{1}{4}\left(-\frac{1}{4}u \cos A\right)$

$= \frac{1}{16}u \cos A$

Second hop:

Initial speed $= \frac{3}{2} u \sin A\vec{i} + \frac{1}{16} u \cos A\vec{j}$

$s_y = 0$

$\Rightarrow \frac{1}{16} u \cos At - \frac{1}{2}g \cos At^2 = 0$

$\Rightarrow t = 0$ **OR** $t = \frac{u}{8g}$

At $t = \frac{u}{8g}$

$s_x = \frac{3}{2}u \sin A\, t + \frac{1}{2}g \sin At^2$

$= \frac{3}{2} u \sin A\left(\frac{u}{8g}\right) + \frac{1}{2}g \sin A\left(\frac{u^2}{64g^2}\right)$

$= \frac{25u^2 \sin A}{128g}$

Q. 4. $v_x = 13u\left(\frac{12}{13}\right) - gt \sin 45°$

$= 12u - \frac{gt}{\sqrt{2}}$

$v_y = 13u\left(\frac{5}{13}\right) - gt \cos 45°$

$= 5u - \frac{gt}{\sqrt{2}}$

$s_x = 13ut\left(\frac{12}{13}\right) - \frac{1}{2}gt^2 \sin 45°$

$= 12ut - \frac{gt^2}{2\sqrt{2}}$

$s_y = 13ut\left(\frac{5}{13}\right) - \frac{1}{2}gt^2 \cos 45°$

$= 5ut - \frac{gt^2}{2\sqrt{2}}$

Need to find landing velocity, i.e. need to find v_x and v_y when $s_y = 0$

$5ut - \frac{gt^2}{2\sqrt{2}} = 0$

$\Rightarrow 10ut\sqrt{2} - gt^2 = 0$

$\Rightarrow t(10u\sqrt{2} - gt) = 0$

$\Rightarrow \underbrace{t = 0}_{\text{Point of Projection}} \qquad \underbrace{t = \frac{10u\sqrt{2}}{g}}_{\text{Time of Flight}}$

$v_x = 12u - \frac{g}{\sqrt{2}}\left[\frac{10u\sqrt{2}}{g}\right]$

$= 12u - 10u$

$= 2u$

$v_y = 5u - \frac{g}{\sqrt{2}}\left[\frac{10u\sqrt{2}}{g}\right]$

$= 5u - 10u$

$= -5u$

$\Rightarrow$ Velocity at landing $= 2u\vec{i} - 5u\vec{j}$

$\vec{j}$-velocity after impact $= -5(-e)$

$= 5e = 5\left(\frac{2}{5}\right)$

$= 2$

$\Rightarrow$ Velocity after impact $= 2u\vec{i} + 2u\vec{j}$

$\Rightarrow$ Particle rebounds at an angle of 45° to the inclined plane. Given that the plane is inclined at 45° to the horizontal, the particle rises vertically (at 90° to horizontal) from P.

Q. 5. (i) $v_x = u\cos 60° + gt\sin 30°$

$= \frac{u}{2} + \frac{gt}{2}$

$v_y = u\sin 60° - gt\cos 30°$

$= \frac{u\sqrt{3}}{2} - \frac{gt\sqrt{3}}{2}$

$s_x = ut\cos 60° + \frac{1}{2}gt^2\sin 30°$

$= \frac{ut}{2} - \frac{gt^2}{4}$

$s_y = ut\sin 60° - \frac{1}{2}gt^2\cos 30°$

$= \frac{ut\sqrt{3}}{2} - \frac{gt^2\sqrt{3}}{4}$

Range: s_x when $s_y = 0$

$\frac{ut\sqrt{3}}{2} - \frac{gt^2\sqrt{3}}{4} = 0$

$\Rightarrow 2ut - gt^2 = 0$

$\Rightarrow t(2u - gt) = 0$

$\Rightarrow t = 0$ (Point of Projection) $\quad t = \frac{2u}{g}$ (Time of Flight)

Range $= \frac{u}{2}\left[\frac{2u}{g}\right] + \frac{g}{4}\left[\frac{4u^2}{g^2}\right]$

$= \frac{u^2}{g} + \frac{u^2}{g} = \frac{2u^2}{g}$

(ii) Need to firstly find landing velocity, i.e. v_x and v_y when $t = \frac{2u}{g}$

$v_x = \frac{u}{2} + \frac{g}{2}\left[\frac{2u}{g}\right] = \frac{u}{2} + u = \frac{3u}{2}$

$v_y = \frac{u\sqrt{3}}{2} - \frac{g\sqrt{3}}{2}\left[\frac{2u}{g}\right] = \frac{u\sqrt{3}}{2} - u\sqrt{3}$

$= -\frac{u\sqrt{3}}{2}$

x–velocity is unchanged after impact

New y–velocity $= \frac{eu\sqrt{3}}{2}$

$\Rightarrow$ velocity at start of 2nd hop

$= \frac{3u}{2}\vec{i} + \frac{eu\sqrt{3}}{2}\vec{j}$

Magnitude $= \sqrt{\frac{9u^2}{4} + \frac{3e^2u^2}{4}}$

$= \frac{u\sqrt{3}}{2}\sqrt{e^2 + 3}$

Let θ be the angle at which the particle leaves the slope

$\tan\theta = \frac{eu\sqrt{3}}{2} \times \frac{2}{3u} = \frac{e\sqrt{3}}{3} = \frac{e}{\sqrt{3}}$

$\Rightarrow \sin\theta = \frac{e}{\sqrt{e^2 + 3}}$ and $\cos\theta = \frac{\sqrt{3}}{\sqrt{e^2 + 3}}$

Projectile equations are:

$v_x = \left[\frac{u\sqrt{3}}{2}\sqrt{e^2 + 3}\right]\left[\frac{\sqrt{3}}{\sqrt{e^2 + 3}}\right]$

$+ gt\sin 30°$

$= \frac{3u}{2} + \frac{gt}{2}$

$v_y = \left[\frac{u\sqrt{3}}{2}\sqrt{e^2 + 3}\right]\left[\frac{e}{\sqrt{e^2 + 3}}\right]$

$- gt\cos 30°$

$= \frac{eu\sqrt{3}}{2} - \frac{gt\sqrt{3}}{2}$

$s_x = \left[\frac{u\sqrt{3}}{2}\sqrt{e^2 + 3}\right][t]\left[\frac{\sqrt{3}}{\sqrt{e^2 + 3}}\right]$

$+ \frac{1}{2}gt^2\sin 30°$

$= \frac{3ut}{2} + \frac{gt^2}{4}$

$s_y = \left[\frac{u\sqrt{3}}{2}\sqrt{e^2 + 3}\right][t]\left[\frac{e}{\sqrt{e^2 + 3}}\right]$

$- \frac{1}{2}gt^2\cos 30°$

$= \frac{eut\sqrt{3}}{2} - \frac{gt^2\sqrt{3}}{4}$

Range: s_x when $s_y = 0$

$\frac{eut\sqrt{3}}{2} - \frac{gt^2\sqrt{3}}{4} = 0$

$\Rightarrow 2eut - gt^2 = 0$

$\Rightarrow t(2eu - gt) = 0$

$t = 0$ (Point of Projection) $\quad t = \frac{2eu}{g}$ (Time of Flight)

Range $= \frac{3u}{2}\left[\frac{2eu}{g}\right] + \frac{g}{4}\left[\frac{4e^2u^2}{g^2}\right]$

$= \frac{3eu^2}{g} + \frac{e^2u^2}{g}$

$= \frac{eu^2}{g}[3 + e]$

Range for 2nd hop = 2[Range for 1st hop]

$$\Rightarrow \frac{eu^2}{g}[3 + e] = 2\left[\frac{2u^2}{g}\right] \quad \ldots \text{ multiply by } \frac{g}{u^2}$$

$$\Rightarrow e(3 + e) = 4$$

$$\Rightarrow 3e + e^2 = 4$$

$$\Rightarrow e^2 + 3e - 4 = 0$$

$$\Rightarrow (e + 4)(e - 1) = 0$$

$$\Rightarrow e \cancel{= -4} \quad e = 1 \quad \ldots 0 \leqslant e \leqslant 1$$

Q. 6. Fall from height h:

$u = 0, \quad a = g, \quad s = h$

$$v = \sqrt{u^2 + 2as} = \sqrt{2gh}$$

$\Rightarrow$ Strikes the plane at a speed of $\sqrt{2gh}$

x-component of velocity at impact

$$= \sqrt{2gh}\cos 60° = \frac{\sqrt{2gh}}{2} = \frac{\sqrt{gh}}{2}$$

y-component of velocity at impact

$$= -\sqrt{2gh}\sin 60° = -\sqrt{2gh}\left(\frac{\sqrt{3}}{2}\right) = -\sqrt{\frac{3gh}{2}}$$

x-component is unchanged during impact

y-component after impact $= \frac{1}{2}\sqrt{\frac{3gh}{2}}$

Magnitude of velocity after impact

$$= \sqrt{\frac{gh}{2} + \frac{3gh}{8}} = \sqrt{\frac{4gh + 3gh}{8}}$$

$$= \sqrt{\frac{7gh}{8}}$$

Let θ be the angle at which the particle leaves the plane

$$\tan\theta = \frac{1}{2}\sqrt{\frac{3gh}{2}} \times \sqrt{\frac{2}{gh}}$$

$$= \frac{\sqrt{3}}{2}$$

$$\Rightarrow \sin\theta = \frac{\sqrt{3}}{\sqrt{7}} \text{ and } \cos\theta = \frac{2}{\sqrt{7}}$$

Projectile equations are:

$$v_x = \left[\sqrt{\frac{7gh}{8}}\right]\left[\frac{2}{\sqrt{7}}\right] + gt\sin 30°$$

$$= \sqrt{\frac{gh}{2}} + \frac{gt}{2}$$

$$v_y = \left[\sqrt{\frac{7gh}{8}}\right]\left[\frac{\sqrt{3}}{\sqrt{7}}\right] - gt\cos 30°$$

$$= \sqrt{\frac{3gh}{8}} - \frac{gt\sqrt{3}}{2}$$

$$s_x = \left[\sqrt{\frac{7gh}{8}}\right][t]\left[\frac{2}{\sqrt{7}}\right] + \frac{1}{2}gt^2\sin 30°$$

$$= t\sqrt{\frac{gh}{2}} + \frac{gt^2}{4}$$

$$s_y = \left[\sqrt{\frac{7gh}{8}}\right][t]\left[\frac{\sqrt{3}}{\sqrt{7}}\right] - \frac{1}{2}gt^2\cos 30°$$

$$= t\sqrt{\frac{3gh}{8}} - \frac{gt^2\sqrt{3}}{4}$$

Range: s_x when $s_y = 0$

$$t\sqrt{\frac{3gh}{8}} - \frac{gt^2\sqrt{3}}{4} = 0 \quad \ldots \text{ multiply by } \frac{4\sqrt{8}}{\sqrt{3}}$$

$$\Rightarrow 4t\sqrt{gh} - gt^2\sqrt{8} = 0$$

$$\Rightarrow 4t\sqrt{gh} - 2gt^2\sqrt{2} = 0$$

$$\Rightarrow 2t\sqrt{gh} - gt^2\sqrt{2} = 0$$

$$\Rightarrow t\sqrt{2gh} - gt^2 = 0$$

$$\Rightarrow t\left(\sqrt{2gh} - gt\right) = 0$$

$$\Rightarrow \underbrace{t = 0}_{\text{Point of Projection}} \qquad \underbrace{t = \frac{\sqrt{2gh}}{g} = \sqrt{\frac{2h}{g}}}_{\text{Time of Flight}}$$

$$\text{Range} = \left[\sqrt{\frac{2h}{g}}\right]\sqrt{\frac{gh}{2}} + \frac{g}{4}\left[\frac{2h}{g}\right]$$

$$\Rightarrow \text{Range} = h + \frac{h}{2}$$

$$= \frac{3h}{2}$$